Geometrical aspects of positional representations of real and complex numbers

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Outline

1. Various approaches to positional representations
2. Rauzy fractals and multiple tilings
3. Purely periodic Rényi expansions
4. Spectra of complex numbers

Positional representations

$$(500_{\bullet})_{10} = (???_{\bullet})_8$$

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2 ways how to compute this:

Division algorithm

Greedy algorithm

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- $500 = 8 \cdot 62 + 4$

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$$(500_\bullet)_{10} = (764_\bullet)_8$$

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Generalizations:

→ Polynomial bases
(“Canonical number systems”)

→ β-expansions

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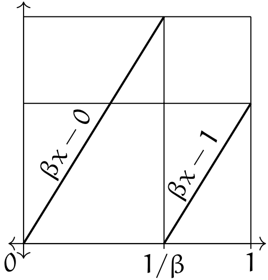
$T_{\text{greedy}} : x \mapsto \beta x - \lfloor \beta x \rfloor$



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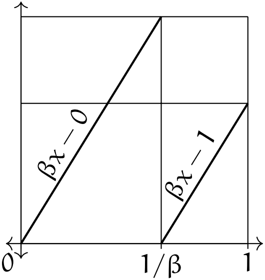

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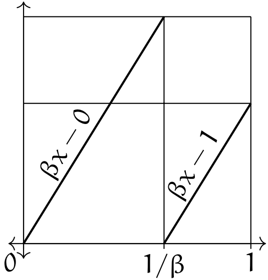

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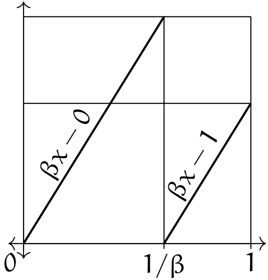

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- Alphabet $\{0, 1, \ldots, \lfloor \beta \rfloor\}$
- Not every string is a β-expansion (there is never 11 for $\beta = \frac{1+\sqrt{5}}{2}$)

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- Algebraic integer: root of (irreducible) polynomial $X^d - a_{d-1}X^{d-1} - a_{d-2}X^{d-2} - \cdots - a_1X - a_0 = 0$ with $a_j \in \mathbb{Z}$

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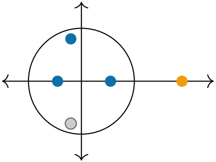
- Pisot number: 

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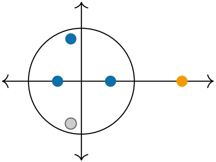
- Pisot number: 
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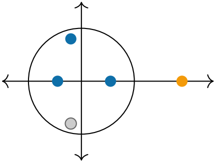
- Pisot number: 
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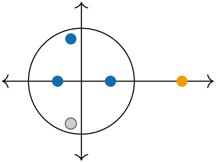
- Pisot number: 
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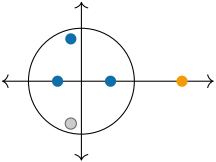
- Pisot number: 
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- Map $\Psi\colon \sum x_j \beta^j \mapsto \left(\sum x_j \beta_{(1)}^j, \ldots, \sum x_j \beta_{(e)}^j\right) \in \mathbb{R}^{d-1}$

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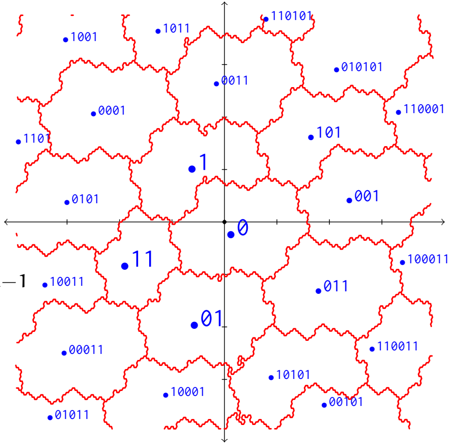

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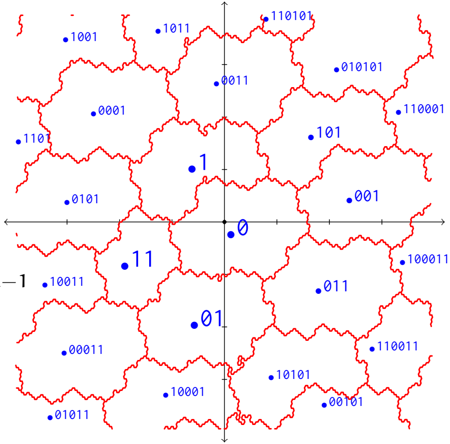

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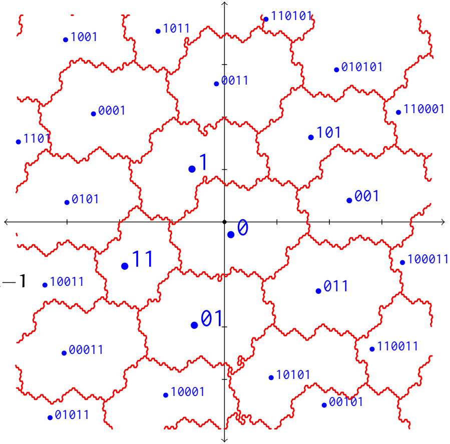

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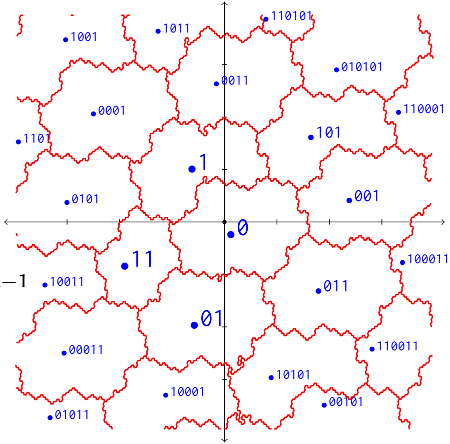

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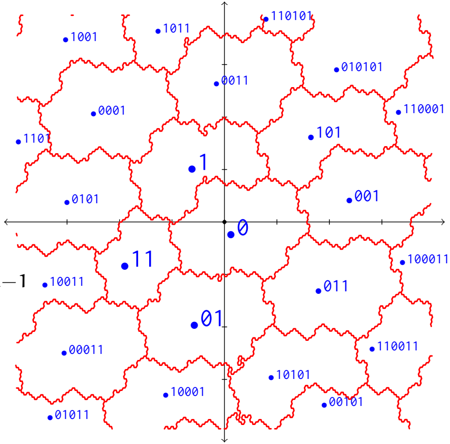

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- Property (W) $\iff$ tiling

Pisot conjecture

1. **For β-numeration:** *Let β be a degree d Pisot unit. Then the collection of Rauzy fractals for the greedy β-transformation is a tiling of $\mathbb{R}^{d-1}$.*
2. **For irreducible substitutions:** *For any Pisot irreducible substitution, the Rauzy fractals form a tiling.*

The two are connected, for instance:

$$\begin{gathered}\beta^3 - \beta^2 - \beta - 1 = 0 \\ \beta \approx 1.839\cdots\end{gathered} \qquad \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \qquad \begin{cases}0 \mapsto 01\\ 1 \mapsto 02\\ 2 \mapsto 0\end{cases}$$

$$0 \mapsto 01 \mapsto 0102 \mapsto 0102010 \mapsto \cdots \mapsto 0102010010201\cdots$$

Theorem (Barge 2015)

The Pisot conjecture for β-numeration is true.

Symmetric β-expansions

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- $T_{\text{greedy}} : [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor$
- $T_{\text{symmetric}} : [-\frac{1}{2}, \frac{1}{2}) \to [-\frac{1}{2}, \frac{1}{2}),\ x \mapsto \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$

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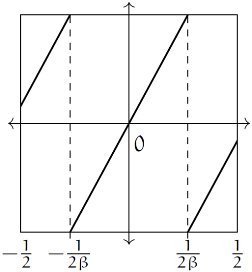

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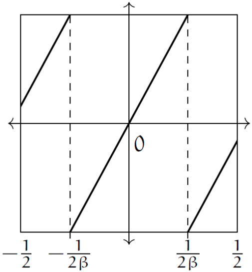
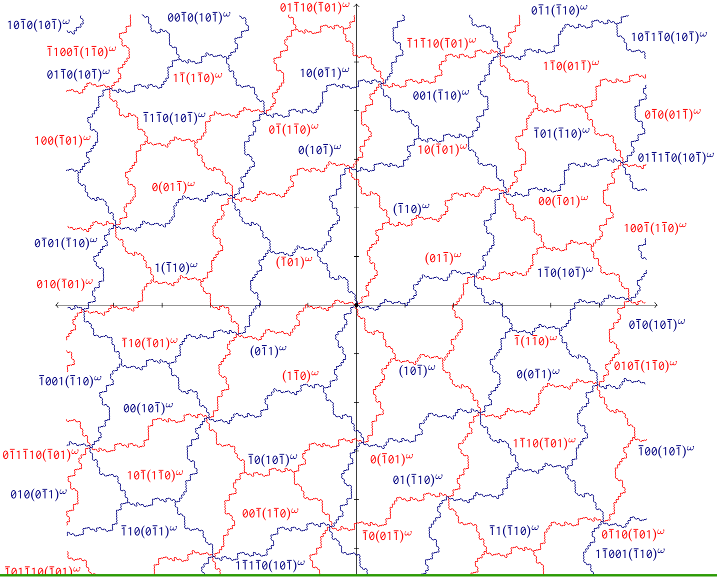


(case $\beta \in (1,2)$)

- Rauzy fractals: defined for each $x \in \mathbb{Z}[\beta] \cap [-\frac{1}{2}, \frac{1}{2})$

Example for symmetric transformation: $\beta^3 - \beta^2 - \beta - 1 = 0$



Degree of the multiple tiling

Theorem (H)

Suppose $\beta \in (1,2)$ is a Pisot unit. Then:

the symmetric β-transformation induces a tiling

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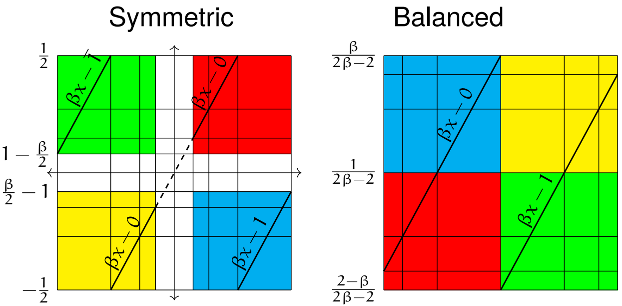
- The case $\beta > 2$ needs different approach
- Symmetric shift radix systems look promising

Core ideas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Balanced
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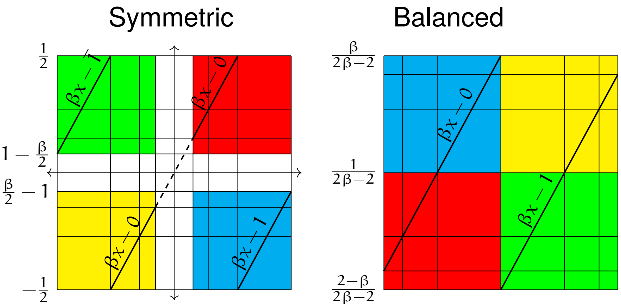

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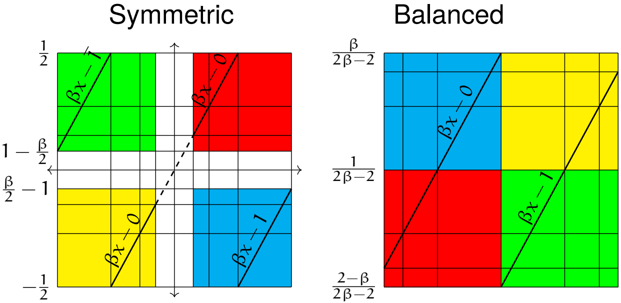

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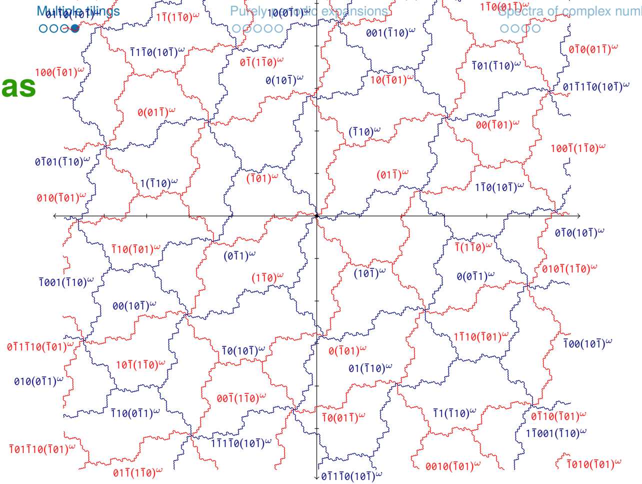

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- Let $\beta^2 = a\beta + 1$, $a \geqslant 1$. Then all $p/q \in \mathrm{Pur}(\beta)$ [Schmidt]

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- Let $\beta^3 = \beta + 1$. Then $p/q \in [0, 0.666666666086\cdots) \implies p/q \in \mathrm{Pur}(\beta)$ [Akiyama, Scheicher]

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$$\gamma(\beta) = \sup \left\{ c > 0 : \text{all } p/q \in [0, c) \text{ such that } q \perp N(\beta) \text{ have a purely periodic expansion} \right\}$$

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- $\gamma(\beta) = 0.999999999999826\cdots$ for $\beta^2 = 56\beta + 8$ [H, Steiner]

The quadratic Pisot case

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- *When* $a = b \geqslant 3$, *we have* $\gamma(\beta) = 0$.
- *In all cases, we can calculate* $\gamma(\beta)$ *with arbitrary precision.*

How do we compute $\gamma(\beta)$?

Natural extension of the greedy β-transformation:

- An a.e.-invertible map $S : \mathrm{NE} \to \mathrm{NE}$ such that

$$\begin{array}{ccc} \mathrm{NE} & \xrightarrow[\text{invertible}]{S} & \mathrm{NE} \\ \downarrow & & \downarrow \\ [0,1) & \xrightarrow{T} & [0,1) \end{array}$$

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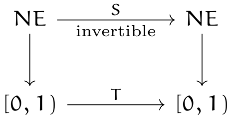

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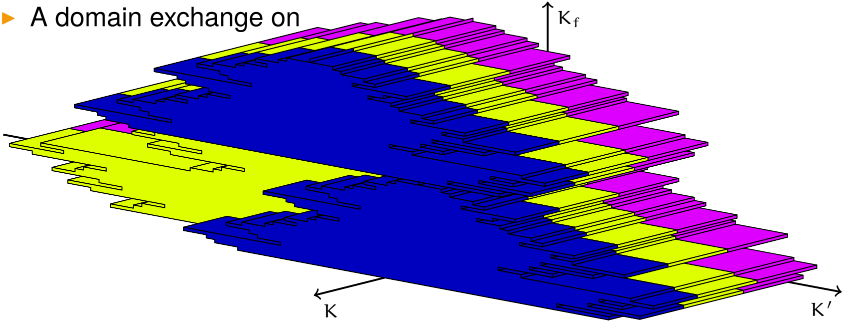
- A domain exchange on



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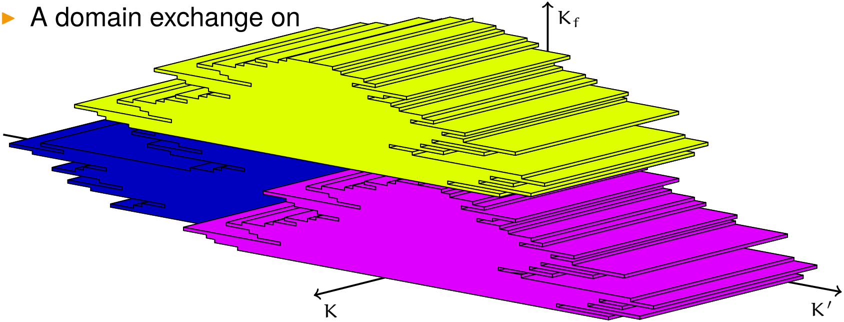

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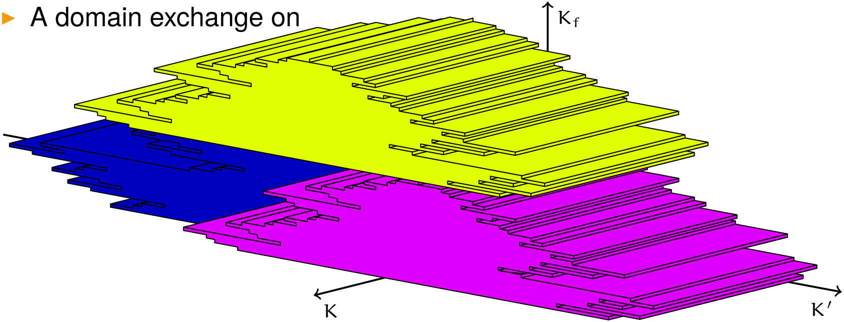
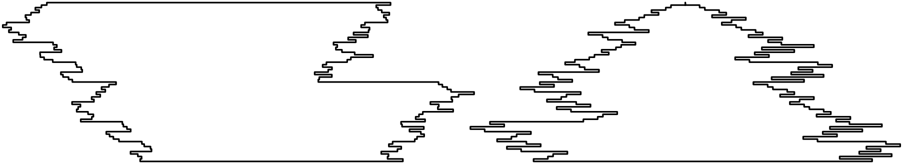


- A point $x \in \mathbb{Q}(\beta)$ has a purely periodic expansion $\iff$ $(x, x', x_f) \in \mathrm{NE}$

The difficulty: Distribution of rational points in the NE

- $\beta^2 = a\beta + b$



The difficulty: Distribution of rational points in the NE

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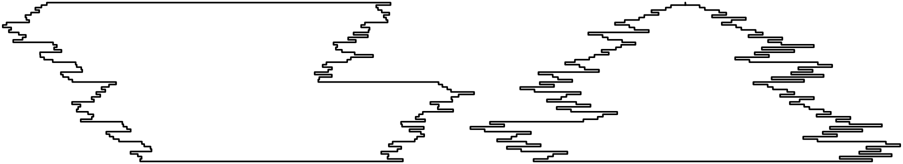

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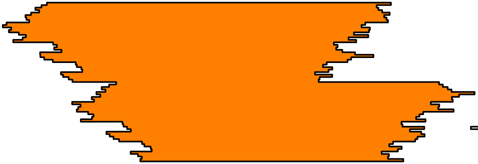
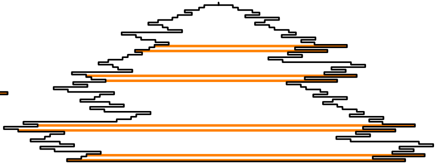


General case



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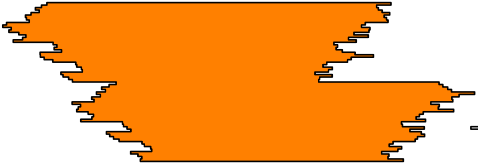

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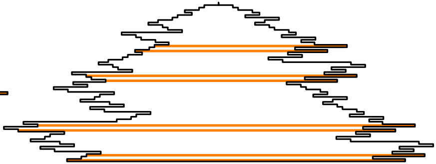


- Description of the boundary [Minervino, Steiner]
- Localization of rational numbers [H, Steiner]

Spectra of numbers

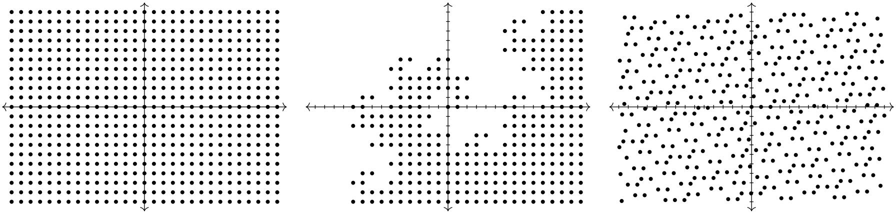
- Spectrum of β with alphabet $\mathcal{A}$: Set of all

$$x_0 + x_1\beta + x_2\beta^2 + \cdots + x_n\beta^n$$

with $x_j \in \mathcal{A}$.

- Restrictions: $|\beta| > 1$ and $0 \in \mathcal{A}$

$i-1, \{0,1\}$ $\quad$ $i+1, \{0,1\}$ $\quad$ $\beta^3 = -\beta^2 - \beta + 1, \{0,1\}$



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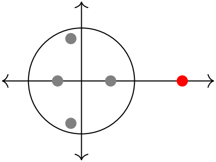
- Motivation:
  - Real: Spectra provide information on the number of different β-representations
  - Complex: The definition of "standard complex β-transformation" is not at all settled down [Hama, Furukado, Ito] [Akiyama, Caalim] [Komornik, Loreti]

The importance of being (complex) Pisot

- Base: β real > 1
- Alphabet: $\mathcal{A} = \{0, 1, \ldots, m\}$

	β Pisot	β non-Pisot
$m + 1 \ggg \beta$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m + 1 > \beta$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m + 1 < \beta$	uniformly discrete not relatively dense	uniformly discrete not relatively dense



[Erdős, Joó, Komornik, Loreti, Pedicini, Bugeaud, Feng, Wen, Borwein, Hare]

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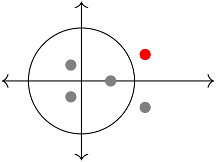
	β complex Pisot	β complex non-Pisot
$m + 1 \ggg \lvert\beta\rvert^2$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m + 1 > \lvert\beta\rvert^2$	uniformly discrete ???	not uniformly discrete ???
$m + 1 < \lvert\beta\rvert^2$	uniformly discrete ???	uniformly discrete ???



[Zaïmi]

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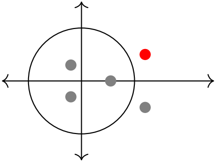
	β complex Pisot	β complex non-Pisot
$m + 1 \ggg |\beta|^2$	**uniformly discrete** **relatively dense**	**not uniformly discrete** **relatively dense**
$m + 1 > |\beta|^2$	**uniformly discrete** **(relatively dense)**	**not uniformly discrete** **???**
$m + 1 < |\beta|^2$	**uniformly discrete** **not relatively dense**	**uniformly discrete** **not relatively dense**



[Zaïmi, H, Pelantová]

Relative denseness

Theorem

Let $\beta \in \mathbb{R}$ such that $|\beta| > 1$.

Suppose $\mathcal{A} = \{0, 1, \ldots, m\}$ with $m + 1 < |\beta|$.

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Theorem (H, Pelantová)

Let $\beta \in \mathbb{C} \setminus \mathbb{R}$ such that $|\beta| > 1$.

Suppose $\mathcal{A} \ni 0$ with $< |\beta|^2$ elements.

Then the spectrum of β with alphabet $\mathcal{A}$ is not relatively dense.

Spectral properties of cubic complex Pisot units

Theorem (H, Pelantová)

Suppose β *is a cubic complex Pisot unit such that it has a real conjugate* β' *and* 0 *is an interior point of* $\mathcal{R}(0)$ *(so-called Property F). Let* $m+1 \geqslant |\beta|^2$*. Then the spectrum of* β *with alphabet* $\{0,1,\ldots,m\}$ *is uniformly discrete and relatively dense.*

Morover, there is an algorithm for calculating the minimal distance for all m *at once.*

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Theorem (H, Pelantová)

Let $\beta \approx -0.771 + 1.115i$, $\beta^3 = -\beta^2 - \beta + 1$, let $m \geqslant 1$, and $k := \left\lfloor \frac{\ln \frac{m}{1-|\beta|^{-2}}}{2\ln|\beta|} \right\rfloor$. Then

$$\textit{minimal distance} = |\beta|^{-k}.$$