Multiple tilings for symmetric β-expansions

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Outline

Basic definitions — β-numeration

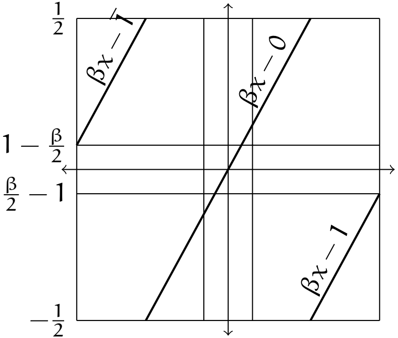
- **Base** $\beta \in (1,2)$, Pisot unit
- d**-Bonacci number**: Pisot root of

$$X^d = X^{d-1} + X^{d-2} + \cdots + X + 1$$

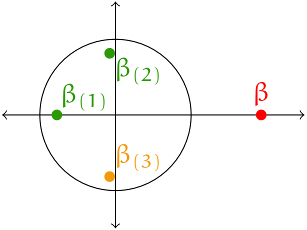
$d = 2$: $\beta = 1.618\cdots$ (Golden ratio)
$d = 3$: $\beta = 1.839\cdots$ (Tribonacci constant)

- **Symmetric β-expansion** [Akiyama, Scheicher, 2007] of $x \in [-\frac{1}{2}, \frac{1}{2})$: the coding of its orbit by

$T_S : X_S \mapsto X_S$



Basic definitions — β-numeration



- **Galois conjugates** $\beta_{(1)}, \dots, \beta_{(e)}$, ignoring those with $\operatorname{Im} \beta_{(j)} < 0$;

$$\Phi\Big(\sum_{\text{finite}} x_j \beta^j\Big) \coloneqq \Big(\sum x_j \beta_{(1)}^j, \dots, \sum x_j \beta_{(e)}^j\Big) \in \text{“}\mathbb{R}^{d-1}\text{”}$$

- **Rauzy fractal** for $x \in \mathbb{Z}[\beta] \cap X_S$:

$$\mathcal{R}(x) \coloneqq \lim_{n\to\infty} \Phi\big(\beta^n T_S^{-n} x\big)$$

Basic definitions — multiple tilings

The collection $\{\mathcal{R}(x)\}_{x \in X_s \cap \mathbb{Z}[\beta]}$ is a **multiple tiling of degree** m

$\Updownarrow$

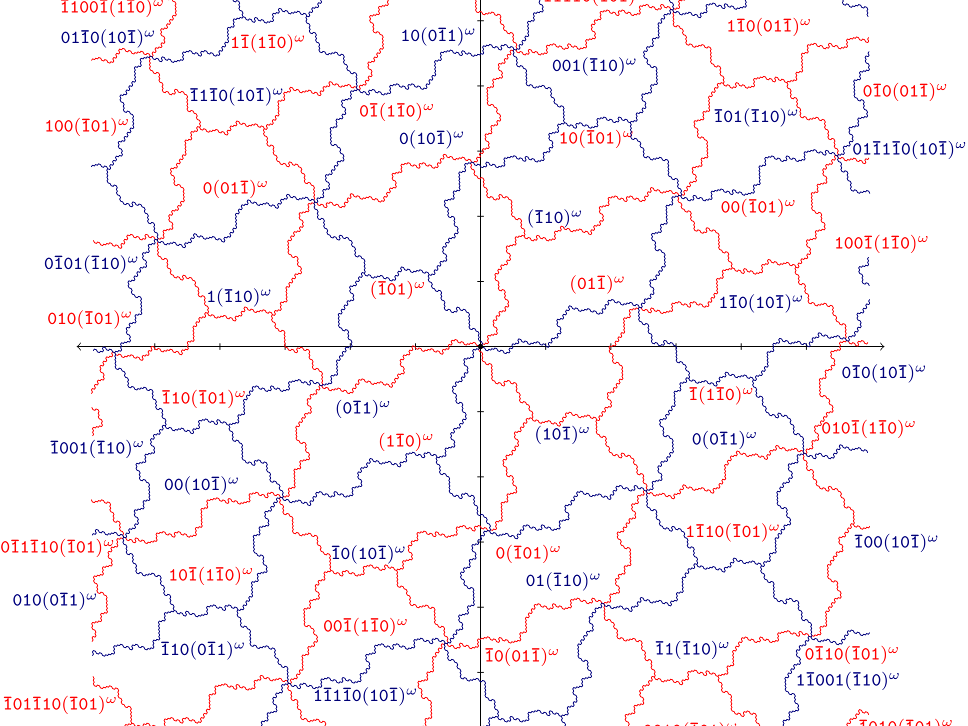
it is a union of m tilings:

- tiles are non-empty, compact, closures of their interior
- each tile has a zero measure boundary
- the collection has finite local complexity
- the collection is locally finite (only finitely many tiles meet any bounded set)
- a.e. point of $\mathbb{R}^{d-1}$ lies in exactly m tiles

Theorem (Kalle, Steiner, 2012)

Suppose β is a Pisot unit. Then for any "well-behaving" β-transformation $T\colon X \mapsto X$, the collection $\{\mathcal{R}(x)\}_{x \in X \cap \mathbb{Z}[\beta]}$ is a multiple tiling.

Pisot conjecture for β-numeration: It is a tiling for greedy expansions
Proved 2 days ago [Barge, arXiv:1505.04408]



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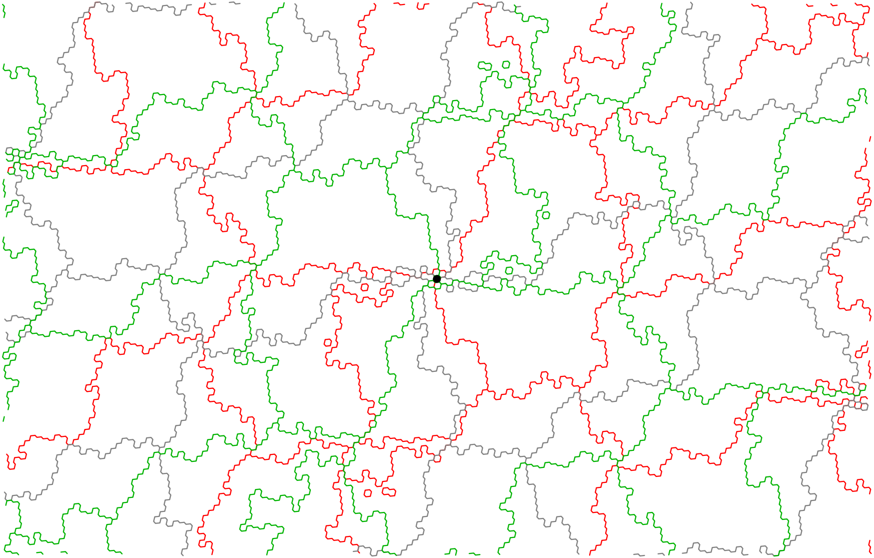

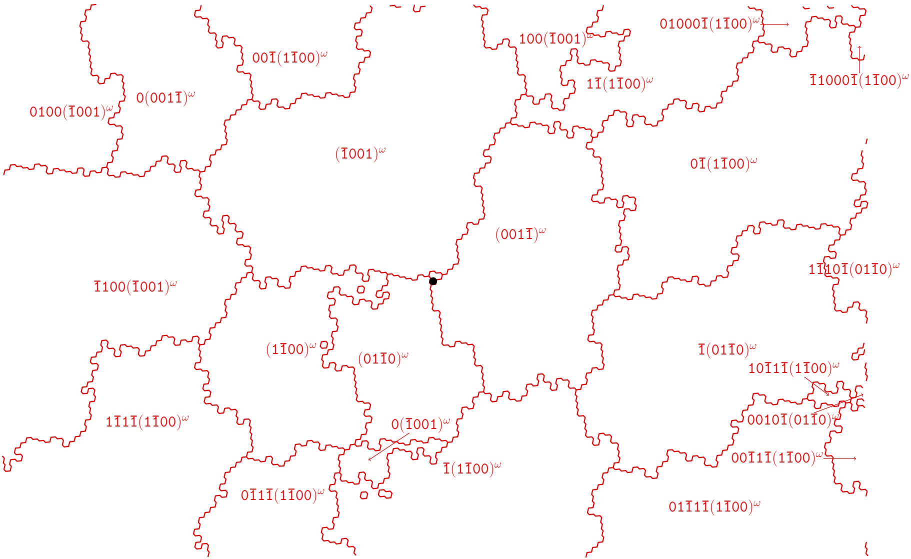

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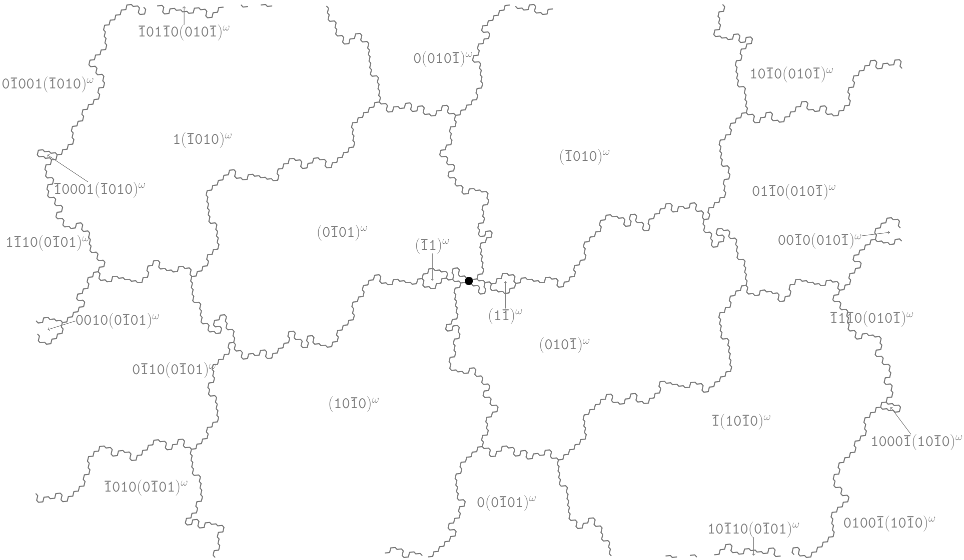
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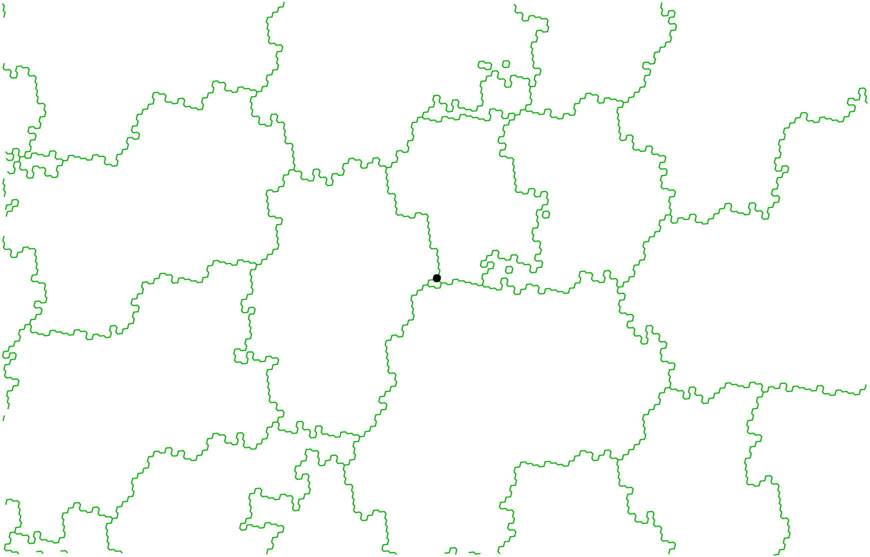


Two main results

Almost-Theorem A

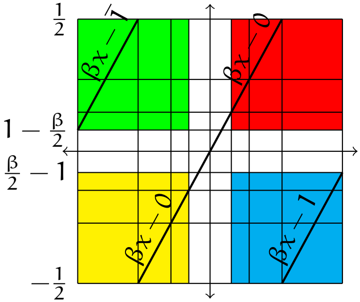
Suppose $\beta \in (1,2)$ *is a Pisot unit. Then* $\{\mathcal{R}(x)\}_{x \in X_S \cap \mathbb{Z}[\beta]}$ *forms a (single) tiling if and only if:*

1. $\beta - 1$ *is a unit; and*
2. $\{\mathcal{R}_B(y)\}_{y \in X_B \cap \mathbb{Z}[\beta]}$ *forms a (single) tiling* *($\mathcal{R}_B$ and X_B explained in the next slide).*

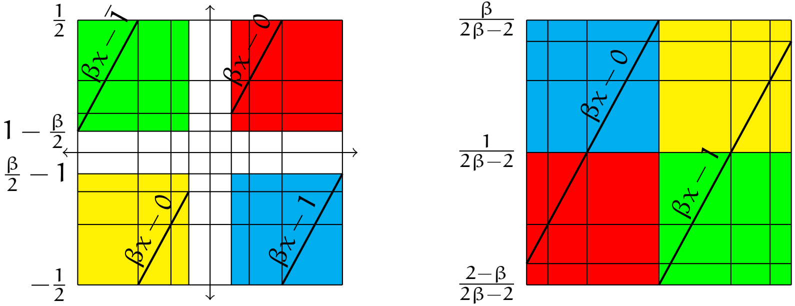
Theorem B

Let $d \geqslant 2$ *and* $\beta^d = \beta^{d-1} + \cdots + \beta + 1$*. Then the symmetric* β*-transformation induces a multiple tiling of* $\mathbb{R}^{d-1}$ *with covering degree* $d - 1$*.*

Symmetric vs. balanced β-transformation



Symmetric vs. balanced β-transformation



symmetric balanced

$$\psi : \quad X_S \longrightarrow X_B$$

$$x \longmapsto \begin{cases} \frac{1}{\beta-1}x & \text{if } x > 0 \\ \frac{1}{\beta-1}(x+1) & \text{if } x < 0 \end{cases}$$

- $T_S(x) = \psi^{-1} T_B \psi(x)$

The idea — Theorem A

Almost-Proposition

Suppose $\beta \in (1,2)$ *is a Pisot unit. Let*

- m_B *be the covering degree of* $\{\mathcal{R}_B(y)\}_{y \in X_B \cap \mathbb{Z}[\beta]}$
- m_S *be the covering degree of* $\{\mathcal{R}_S(x)\}_{x \in X_S \cap \mathbb{Z}[\beta]}$

Then

$$m_S = \left| N(\beta - 1) \right| \cdot m_B$$

$$\mathcal{R}_S(x) \xleftarrow[n\to\infty]{\Phi} \beta^n T_S^{-n} x = \beta^n \psi^{-1} T_B^{-n} \psi x = \beta^n \Big((\beta - 1) T_B^{-n} \psi x - \textbf{?} \Big)$$

$$= (\beta - 1) \beta^n T_B^{-n} \psi x - \beta^n \textbf{?} \xrightarrow[n\to\infty]{\Phi} \Phi(\beta - 1) \times \mathcal{R}_B(\psi x) \quad \text{if} \quad y \coloneqq \psi x \in \mathbb{Z}[\beta]$$

- $\psi x = \frac{1}{\beta - 1}(x + \textbf{?})$
- $\psi^{-1} y = (\beta - 1) y - \textbf{?}$

The idea — Theorem A

Almost-Proposition

Suppose $\beta \in (1,2)$ *is a Pisot unit. Let*

- m_B *be the covering degree of* $\{\mathcal{R}_B(y)\}_{y \in X_B \cap \mathbb{Z}[\beta]}$
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$$= (\beta-1)\beta^n T_B^{-n} \psi x - \beta^n \textbf{?} \xrightarrow[n\to\infty]{\Phi} \Phi(\beta-1) \times \mathcal{R}_B(\psi x) \quad \text{if} \quad y := \psi x \in \mathbb{Z}[\beta]$$

Therefore:

$$\underbrace{\mathcal{R}_S((\beta-1)y \bmod 1))}_{\text{this is } \frac{1}{|N(\beta-1)|} \text{ of all tiles}} = \mathcal{R}_S(\psi^{-1}(y)) = \underbrace{\Phi(\beta-1)}_{\text{regular linear transformation in } \mathbb{R}^{d-1}} \times \mathcal{R}_B(y)$$

The idea — Theorem B

Theorem B

Let $d \geqslant 2$ and $\beta^d = \beta^{d-1} + \cdots + \beta + 1$. Then the symmetric β-transformation induces a multiple tiling of $\mathbb{R}^{d-1}$ with covering degree $d-1$.

- The norm of $\beta - 1$ is $N(\beta - 1) = \pm(d-1)$
- Show that for d-Bonacci numbers, $\{\mathcal{R}_B(y)\}_{y \in X_B \cap \mathbb{Z}[\beta]}$ is a tiling
- We show that all $z \in \mathbb{Z}[\beta]$ satisfy $T_B^k(z) = 1/\beta$ for some k
  (using Property (F) for greedy expansions [Frougny, Solomyak, 1992])
- Arithmetic condition [Kalle, Steiner, 2012]:
  $\Phi(z) \in \mathcal{R}(x) \Leftrightarrow$ there exists $y \in \mathbb{Z}[\beta]$ such that:
  1. $T^p(y) = y$ for some p
  2. $x = T^k(y + \beta^{-k} z)$ for some large enough k
- In the end, it's just addition of 1

Conclusion

- + Strong link between tiling properties for symmetric and balanced expansions.
- + Covering degree specified for symmetric d-Bonacci expansions.
- – Works only for unit $\beta \in (1, 2)$.
- ? Is there a link between the tiling for balanced and greedy transformation? Maybe when $\beta > \frac{1+\sqrt{5}}{2}$?

Thank you for your attention