What is the Abelianization of the Tribonacci shift?

(work in progress)

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The question & the answer

What are the infinite words such that each their factor is a shuffle of a word in the Tribonacci language?

Certainly not only the words in the Tribonacci shift.

More precisely…

- Tribonacci substitution

$$\varphi\colon 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$$

- Tribonacci word

$$\mathbf{t} = \varphi^{\infty}(0) = 0102010\,010201\,0102010\,01020\cdots \in \Sigma^{\omega}$$

- Tribonacci shift

$$\Sigma(\mathbf{t}) \coloneqq \overline{\{\sigma^n(\mathbf{t}) : n \in \mathbb{N}\}} = \{\boldsymbol{w} \in \Sigma^{\omega} : \mathcal{L}(\boldsymbol{w}) \subseteq \mathcal{L}(\mathbf{t})\}$$

- shuffle of $u_0 u_1 \cdots u_{n-1}$ is any word Abelian-equivalent to u
- $\mathcal{A}\mathcal{L}$ is the set of all shuffles of words in language $\mathcal{L}$
- $\mathcal{A}\mathcal{L}(\mathbf{t})$ is the set of all shuffles of Tribonacci factors

Question: Determine $\mathcal{A}\Sigma(\mathbf{t}) \coloneqq \{\boldsymbol{w} \in \Sigma^{\omega} : \mathcal{L}(\boldsymbol{w}) \subseteq \mathcal{A}\mathcal{L}(\mathbf{t})\}$.

Some basic facts

Example: $\mathcal{A}\mathcal{L}(01001) = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100,$
$0001, 0010, 0100, 1000, 0011, 0101, 0110, 1001, 1010, 1100,$
$00011, 00101, 00110, 01001, 01010, 01100, 10001, 10010, 10100, 11000\}$

After factor-closing: $\{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100,$
$0010, 0100, 0101, 1001, 1010,$
$00101, 01001, 01010, 10010, 10100\}$

Observation

$$\begin{aligned} \mathcal{A}\Sigma(\text{Sturmian word}) &= \Sigma(\text{Sturmian word}) \\ \mathcal{A}\Sigma((01)^\omega) &= \Sigma((01)^\omega) = \{(01)^\omega, (10)^\omega\} \\ \mathcal{A}\Sigma((0110)^\omega) &= \Sigma((0110)^\omega) \cup \Sigma((01)^\omega) \\ \mathcal{A}\Sigma(110(01)^\omega) &= \{0, 1, \epsilon\}\{01, 10\}^\omega \end{aligned}$$

Tribonacci word — the naive approach

- $\mathtt{t}$ is a characteristic Arnoux-Rauzy word:
  - complexity $2n+1$
  - one left-special factor of each length that has 3 extensions
- **Left extension branches** — fixed points of φ^3:

$$\left.\begin{array}{r}\cdots 0102010010201001020101020100102\\ \cdots 0010201010201001020102010010201\\ \cdots 0102010010201020100102010102010\end{array}\right\rangle 0102010010201010 2010\cdots$$

- **"Trivial" non-trivial members** of $\mathcal{A}\Sigma(\mathtt{t})$:

$$\begin{array}{rl}
20 & 01020100102010102010\cdots\\
2001 & 01020100102010102010\cdots\\
20010102 & 01020100102010102010\cdots\\
20010201 & 01020100102010102010\cdots\\
200101020102010 & 01020100102010102010\cdots\\
200101020100102 & 01020100102010102010\cdots\\
200102010010201 & 01020100102010102010\cdots\\
200102010102010 & 01020100102010102010\cdots\\
 & \vdots
\end{array}$$

Tribonacci word — we observe that...

- $200101020102010 \quad 010201001020101020 10\cdots$
- $2\ 0\ 01\ 0102\ 0102010\ 0102010010201$
- $2\ 0\ \varphi(0)\ \varphi^2(0)\ \varphi^3(0)\ \varphi^4(0)\cdots =: \mathbf{s}$
- $\mathbf{s} = 2\ \varphi(\mathbf{s})$ and $\mathbf{s}$ is substitutive: $0 \mapsto 01$, $1 \mapsto 02$, $2 \mapsto 0$, $2' \mapsto 2'0$

Theorem

$$\mathbf{s} \in A\Sigma(\mathbf{t})$$

Open question

$$A\Sigma(\mathbf{t}) \stackrel{?}{=} \Sigma(\mathbf{s}) + \{\text{words from previous slide}\}$$

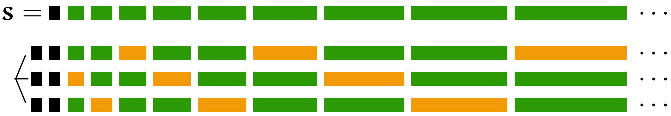
Tribonacci word — proof sketch

- Flip the left extensions:

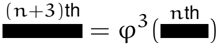
$$s = 20\ 01\ 0102\ 0102010\ 0102010010201\ 0 \cdots$$

0102 01 0102 0100102 0102010010201 0 ⋯
1020 10 0102 0102010 0102010102010 0 ⋯
2010 01 0201 0102010 0102010010201 0 ⋯

- Schematically:



- What's left: Prove that



(involves the incidence matrix, Abelianization, palindromicity, . . .)

More questions than answers — as promised

Tribonacci:

1. Fully characterize $\mathcal{A}\Sigma(\mathbf{t})$
2. We have $\bigcap_n \sigma^n(\Sigma(\mathbf{s})) = \Sigma(\mathbf{t})$; do we also have $\bigcap_n \sigma^n(\mathcal{A}\Sigma(\mathbf{t})) = \Sigma(\mathbf{t})$?
3. Is it true that $\mathrm{L}(\mathbf{t}) \subseteq \mathrm{L}(\boldsymbol{w})$ for any $\boldsymbol{w} \in \mathcal{A}\Sigma(\mathbf{t})$?

General:

4. Which words other than Sturmian satisfy that $\mathcal{A}\Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{w})$?
5. What about other AR words?
   - other fixed points of AR substitutions on 3 letters
   - generic AR words on 3 letters
   - larger alphabets
6. Other classes of words:
   - codings of interval exchange, billiard, …
   - episturmian, rich, …
   - automatic, (purely) morphic words