Purely periodic beta-expansions in quadratic non-unit bases

Tomáš Hejda
joint work with Wolfgang Steiner
with results of Milton Minervino

LIAFA, Univ. Paris Diderot & Doppler Institute, Czech Tech. Univ. in Prague

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Definitions and Goal

Rényi β-expansion of $x \in [0,1)$:

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots \qquad x_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$$

$x_1 x_2 x_3 \cdots$ lexicographically largest possible

Or defined by transformation:

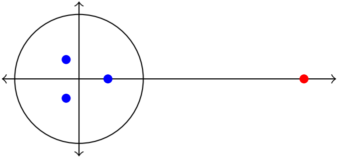
$$x_1 = \lfloor \beta x \rfloor, \qquad T(x) = \beta x - x_1 = \beta x - \lfloor \beta x \rfloor$$

Expandable to all positive numbers:

$$x_{-N} \cdots x_{-3} x_{-2} x_{-1} x_0 . x_1 x_2 x_3 \cdots \qquad \sum_{i \geqslant -N} \frac{x_i}{\beta^i}$$

Definitions and Goal

We restrict to β being a **Pisot number**:



Theorem (Schmidt)

All rational numbers have eventually periodic β-expansion.

Example (Schmidt, Akiyama)

$\beta \approx 1.325$ the smallest Pisot number, root of $\beta^3 = \beta + 1$.

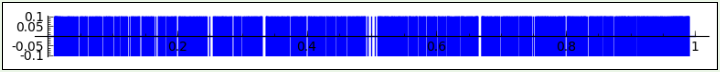
{Picture on whiteboard: all p/q, where $q < 100$, with not purely periodic expansion.}

Definitions and Goal

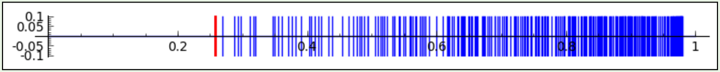
Example (Minervino, Steiner)

$\beta \approx 3.562$ root of $\beta^2 = 3\beta + 2$.

All p/q, where $q < 100$, with not purely periodic expansion:



All p/q, where $q < 100$ and $q \perp 2$, with not purely periodic expansion:



Gamma of beta:

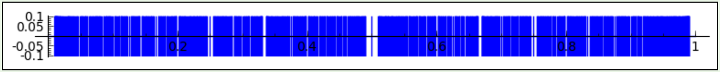
$$\gamma(\beta) := \sup\{c \geqslant 0 : \text{all } p/q \in [0, c),\ q \perp N(\beta), \text{ have pur. per. } \beta\text{-expansion}\}$$
$$= \inf\{p/q \in [0, 1] : q \perp N(\beta) \text{ and } p/q \text{ has not pur. per. } \beta\text{-expansion}\}$$

Definitions and Goal

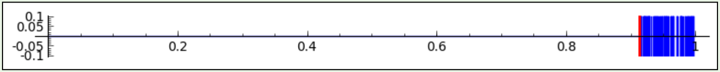
Example (Hejda, Steiner)

$\beta \approx 2.732$ root of $\beta^2 = 2\beta + 2$.

All p/q, where $q < 100$, with not purely periodic expansion:



All p/q, where $q < 10000$ and $q \perp 2$, with not purely periodic expansion:



Gamma of beta:

$$\gamma(\beta) \coloneqq \sup\{c \geqslant 0 : \text{all } p/q \in [0,c),\ q \perp N(\beta), \text{ have pur. per. } \beta\text{-expansion}\}$$
$$= \inf\{p/q \in [0,1] : q \perp N(\beta) \text{ and } p/q \text{ has not pur. per. } \beta\text{-expansion}\}$$

Results

Theorem (Akiyama, Barat, Berthé, Siegel; Minervino, Steiner)

Let β be a Pisot number, root of $\beta^2 = a\beta + b$ with $a \perp b$.

1. *If $a > b(b-1)$ then $\gamma(\beta) = 1 - \frac{(b-1)b\beta}{\beta^2 - b^2}$.*
2. *If $a \leqslant b(b-1)$ then $\gamma(\beta) = 0$.*

Theorem (Hejda, Steiner)

Let β be a Pisot number, root of $\beta^2 = a\beta + b$ with $b \mid a$.

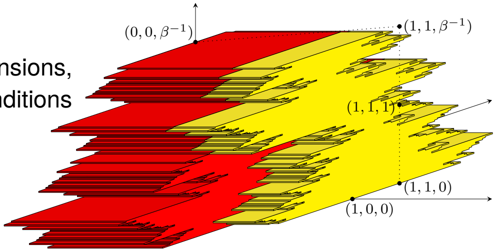
1. *If $a > b(b-1)$ then $\gamma(\beta) = 1$.*
2. *If $a \leqslant b(b-1)$ then $\gamma(\beta)$ can be computed (fast) with arbitrary precision.*

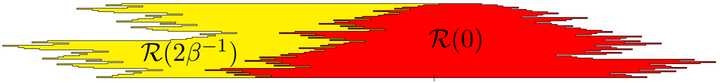
Conjecture

In the last case above, we have $\gamma(\beta) < 1$.

Tools

1. β-adic expansions
   - generalizations of p-adic expansions to any algebraic integer β
2. β-tiles
   - tiles arising in natural extensions, giving good periodicity conditions





Theorem (Hama, Imahashi; Ito, Rao; Berthé, Siegel)

$x \in \mathbb{Q}$ *has purely periodic β-expansion* $\iff$ $(x, x, \ldots, x) \in \mathcal{NE}$.

β-adic expansions

$\mathbb{Z}_p \coloneqq \left\{ \frac{x}{q} : x \in \mathbb{Z}, q \in \mathbb{N}, q \perp p \right\}$. Example: $\frac{3}{5} \in \mathbb{Z}_2$, $\frac{3}{10} \notin \mathbb{Z}_2$

p**-adic expansions**:

- $n_0 \in \mathbb{Z}_p$
- $n_0 = a_0 + n_1 p$ with $n_1 \in \mathbb{Z}_p$
- $n_0 = a_0 + (a_1 + n_2 p)p = a_0 + a_1 p + n_2 p^2$
- $n_0 = \sum_{i=0}^{\infty} a_i p^i$

Example

$p = 2$ and $n_0 = -1$

$-1 = 1 + (-1)2 = 1 + 1 \cdot 2 + (-1) \cdot 2 = \cdots = 1 + 1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + \cdots$

p-adic expansion of -1 is $\cdots 1111111.$

But as well:

p-adic expansion of -1 is $\cdots 0303031.$

β-adic expansions

$\mathbb{Z}_{N(\beta)}[\beta] = \{\frac{x_0 + x_1\beta + x_2\beta^2 + \cdots + x_{d-1}\beta^{d-1}}{q} : x_i \in \mathbb{Z}, q \in \mathbb{N}, q \perp N(\beta)\}$

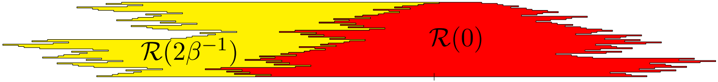
β-adic expansions:

- $n_0 \in \mathbb{Z}_{N(\beta)}[\beta]$
- $n_0 = a_0 + n_1\beta$ with $n_1 \in \mathbb{Z}_{N(\beta)}[\beta]$
- $n_0 = a_0 + (a_1 + n_2\beta)\beta = a_0 + a_1 p + n_2\beta^2$
- $n_0 = \sum_{i=0}^{\infty} a_i\beta^i$

Example

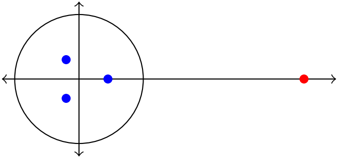
- $\beta = \frac{3+\sqrt{17}}{2}$, root of $\beta^2 = 3\beta + 2$
- Then $N(\beta) = 2$
- β-adic expansion of $n_0 = \frac{1}{1-\beta} = \beta - 4$ is $\cdots 1111111.$

β-tiles



β-tiles $\mathcal{R}(y)$ of $y \in \mathbb{Z}[1/\beta]$:

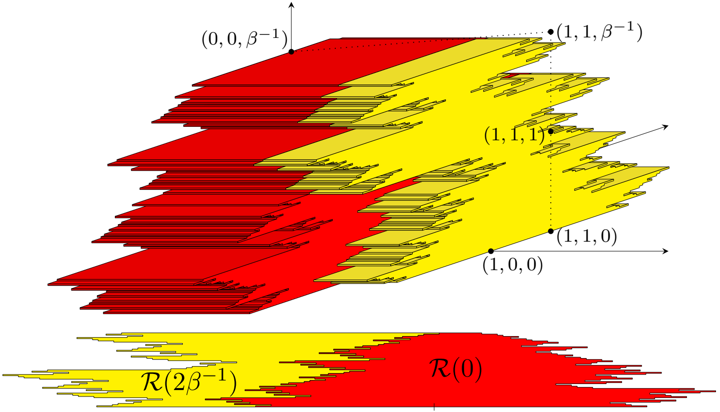
- for fixed $k \in \mathbb{N}$, take all numbers $x \in \beta^k\big(T^{-k}(y)\big)$
- take the conjugates $x' \in \mathbb{Q}(\beta')$



  and embedding of x in the space of β-adic numbers

- do the Hausdorff limit for $k \to \infty$

β-tiles



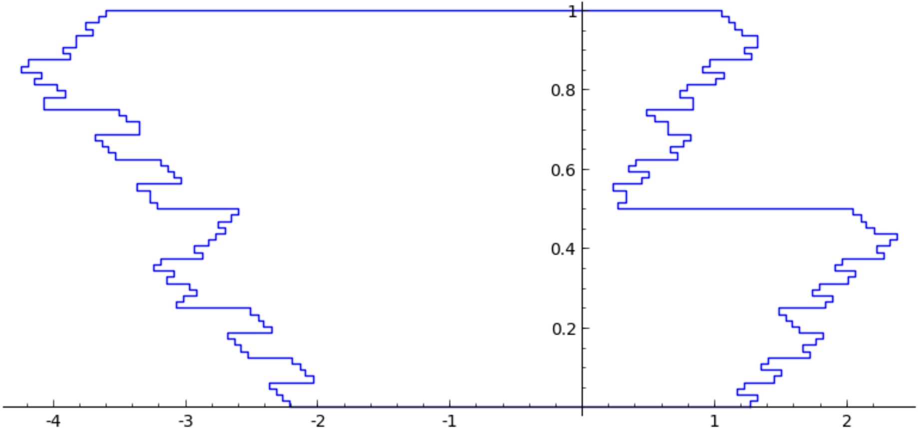
Natural extension:

- $\mathcal{NE}$ is a suspension of two shifted β-tiles:

$$\mathcal{O} - \mathcal{R}(\mathcal{O}) \qquad \text{and} \qquad \{\beta\} - \mathcal{R}(\{\beta\})$$

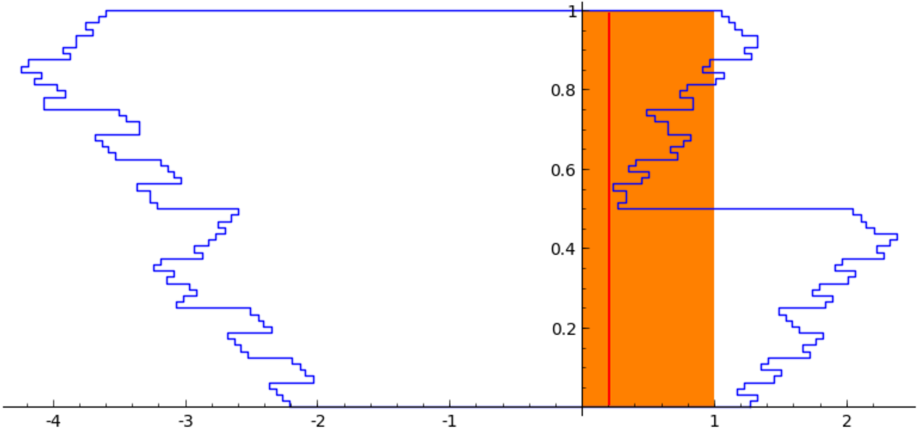
Proof idea — Case $\mathfrak{a} \perp \mathfrak{b}$ [Minervino, Steiner]

- $\beta^2 = 3\beta + 2$



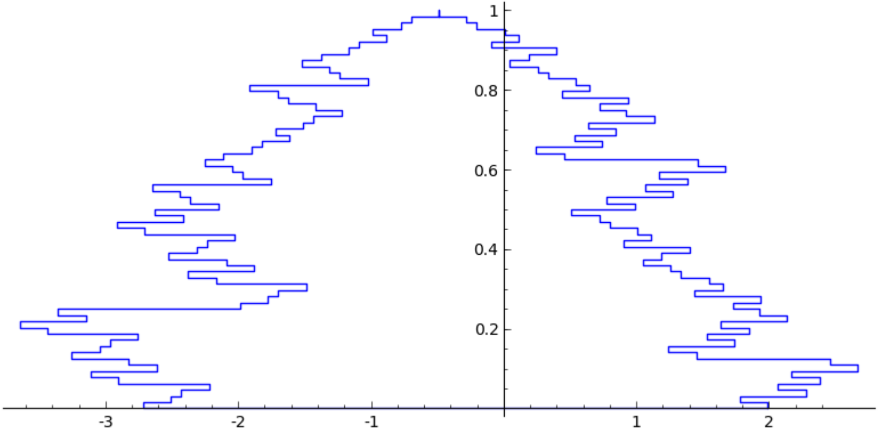
Proof idea — Case $\mathfrak{a} \perp \mathfrak{b}$ [Minervino, Steiner]

- $\beta^2 = 3\beta + 2$
- Where the rational numbers lie:



Proof idea — Case $b \mid a$ [Hejda, Steiner]

- $\beta^2 = 2\beta + 2$



Proof idea — Case $b \mid a$ [Hejda, Steiner]

- $\beta^2 = 2\beta + 2$
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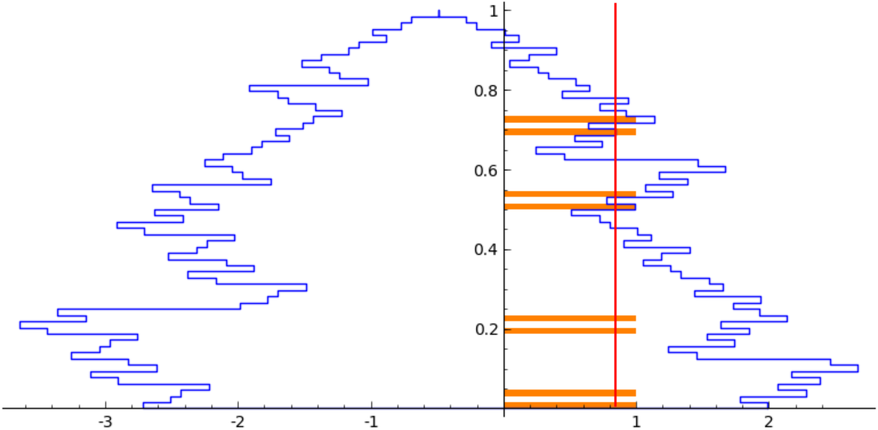


Proof idea – What's left

1. Shape of the boundary:

Proposition (Minervino, Steiner)

The boundary points of the right boundary are

$$\Bigg(\underbrace{\sum_{k \geqslant 0} a_k (\beta')^k}_{x\text{-direction}} \ , \ \underbrace{\sum_{k \geqslant 0} a_k \beta^k}_{\beta\text{-adic direction}} \Bigg)$$

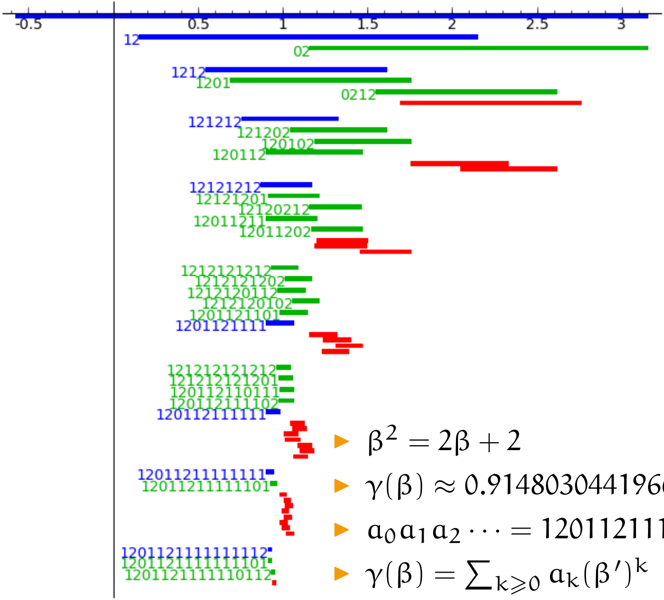
for $a_0 a_1 a_2 a_3 \cdots \in \big(\{0, \ldots, b-1\}\{a-b+1, \ldots, a\}\big)^\omega$.

2. Where do the rational points lie:

Proposition (Hejda, Steiner)

In the case $b \mid a$, the number of prefixes of β-adic expansions of rational points of the lengths $2n-1, 2n$ is b^n.
They can be computed considering all points $i/(b^n+1) \in [0, 1)$.

Algorithm for computing $\gamma(\beta)$



- $\beta^2 = 2\beta + 2$
- $\gamma(\beta) \approx 0.914803044196658195047\cdots$
- $a_0 a_1 a_2 \cdots = 120112111111111211\cdots$
- $\gamma(\beta) = \sum_{k\geqslant 0} a_k (\beta')^k$

Open problems

1. The case $a \not\lfloor b$ but $b \nmid a$? For example $\beta^2 = 6\beta + 4$.
2. In the case $a \mid b$, is $\gamma(\beta)$ in $\mathbb{Q}(\beta)$, or at least algebraic?
3. Higher dimensions? We don't know the boundary description.