Spectral properties of cubic complex Pisot units

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Outline

- Goal: Study the set

$$X^m(\gamma) = \{a_0 + a_1\gamma + a_2\gamma^2 + \cdots + a_n\gamma^n : a_i \in \{0, 1, \ldots, m\}\}$$

  for complex γ

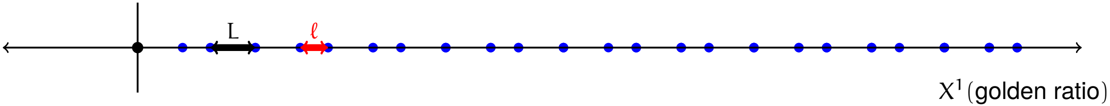
- Results:
  - Strong necessary condition for relative denseness
  - Minimal and maximal distances in $X^m(\gamma)$ for many cases

- Tools:
  - Beta-numeration
  - Cut-and-project sets

Delone sets

- In the real case $\beta > 1$: $X^m(\beta) = \{x_0 = 0 < x_1 < x_2 < \dots\}$ we put:

$$\ell_m(\beta) := \liminf(x_{j+1} - x_j) \qquad L_m(\beta) := \limsup(x_{j+1} - x_j)$$



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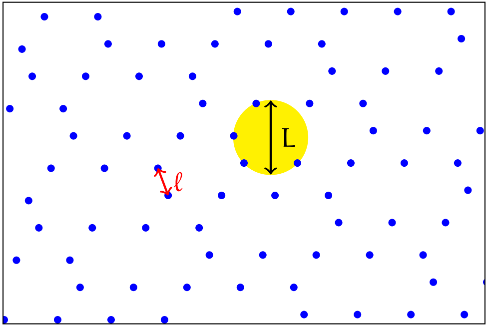
- In the complex case:
  - Uniformly discrete ... $\ell > 0$
  - Relatively dense ... $L < \infty$
  - Delone set ... both



X^2(complex Tribonacci)

Delone sets

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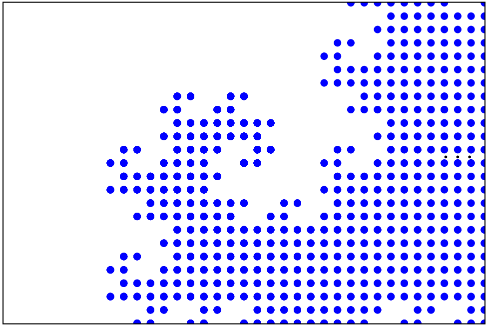
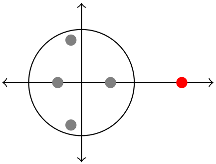


$X^1(1+i)$

Overview – real case $\beta > 1$

- Erdős, Joó, Komornik, Loreti, Pedicini, Bugeaud, Feng, Wen, Borwein, Hare, ...
  The set $X^m(\beta)$ is:

	β Pisot	β non-Pisot
$m+1 \ggg \beta$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m+1 > \beta$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m+1 < \beta$	uniformly discrete not relatively dense	uniformly discrete not relatively dense

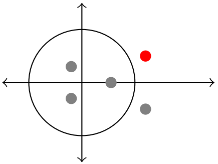


Overview – complex case

- Zaïmi

The set $X^m(\gamma)$ is:

	γ complex Pisot	γ complex non-Pisot
$m+1 \ggg \|\gamma\|^2$	**uniformly discrete** **relatively dense**	**not uniformly discrete** **relatively dense**
$m+1 > \|\gamma\|^2$	**uniformly discrete** **???**	**not uniformly discrete** **???**
$m+1 < \|\gamma\|^2$	**uniformly discrete** **???**	**uniformly discrete** **???**

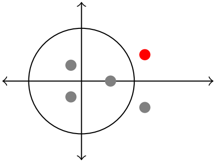


Results I

- Zaïmi, Hejda, Pelantová

The set $X^m(\gamma)$ is:

	γ complex Pisot	γ complex non-Pisot
$m+1 \ggg \|\gamma\|^2$	uniformly discrete relatively dense	not uniformly discrete relatively dense
$m+1 > \|\gamma\|^2$	uniformly discrete (relatively dense)	not uniformly discrete ???
$m+1 < \|\gamma\|^2$	uniformly discrete not relatively dense	uniformly discrete not relatively dense



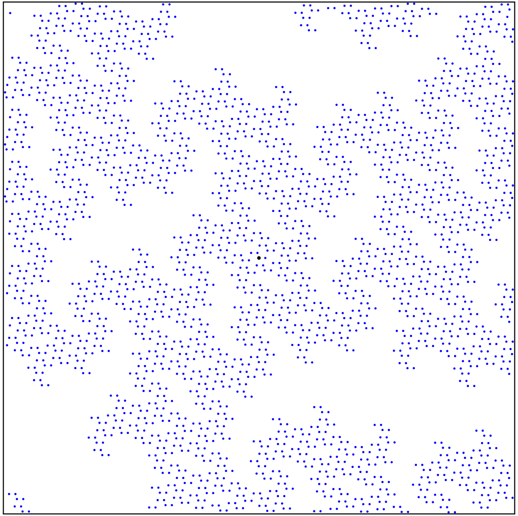
Results II

Theorem

Let γ be non-real, $|\gamma| > 1$. Let $m \in \mathbb{N}$ be such that $m + 1 < |\gamma|^2$. Then $X^m(\gamma)$ is not relatively dense.

Proof ideas:

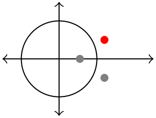
- $f(r) := \#\big(X^m(\gamma) \cap B(0, r)\big)$
- We show that $f(|\gamma|r - m) \leqslant (m + 1)f(r)$
- Therefore $f(r)/r^2 \to 0$ when $r \to \infty$

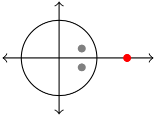


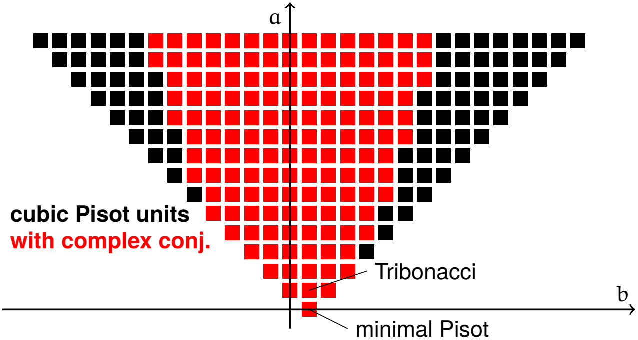
$X^1(\gamma), \quad \gamma^3 = -2\gamma^2 - \gamma + 1, \quad |\gamma|^2 \approx 2.147$

Cubic complex Pisot numbers

- $\gamma \in \mathbb{C} \setminus \mathbb{R}$ is cubic complex Pisot unit:
  - root of $X^3 + bX^2 + aX - 1$, $a, b \in \mathbb{Z}$ such that $|\gamma| > 1$
  - three roots: $\gamma \in \mathbb{C}$, $\overline{\gamma} \in \mathbb{C}$, $\gamma' \in \mathbb{R}$
  - $0 < \gamma' < 1$
  - $\beta := 1/\gamma'$ is a real Pisot number, root of $X^3 - aX^2 - bX - 1$

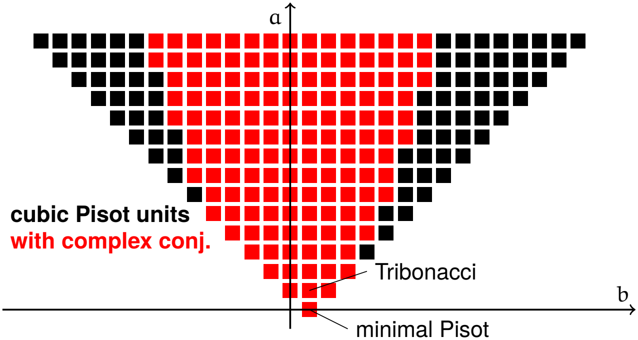






Beta-numeration

- β-representation of $x \in \mathbb{R}_+$:
  - a string $a_N \dots a_0 \bullet a_{-1} \dots$, with $a_j \in \mathbb{N}$ such that $\sum a_j \beta^j = x$
  - Rényi expansion: the lexicographically largest β-representation
  - β satisfies Property (F) if all $x \in \mathbb{Z}[\beta]$ have finite Rényi expansion
  - Frougny, Solomyak, Akiyama: description of cubic units with (F)



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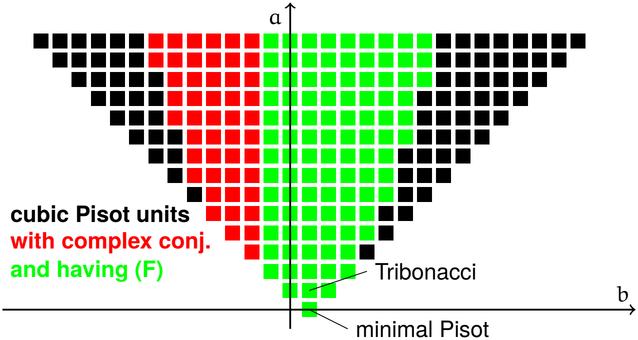

Cut-and-project sets

Theorem

Let γ be a CCPU and suppose $\beta := 1/\gamma'$ satisfies (F). Let $m \geqslant \beta$ and $x \in \mathbb{Z}[\gamma]$. Then the following statements are equivalent:

1. $x \in X^m(\gamma)$
2. x' *has a finite* β*-representation* $a_0 \bullet a_1 \cdots a_k$ *with* $0 \leqslant a_j \leqslant m$
3. $x' \in \big[0, m\beta/(\beta - 1)\big)$

- $1 \Leftrightarrow 2$ elementary
- $2 \Rightarrow 3$ elementary

Cut-and-project sets

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- $3 \Rightarrow 2$:

$$x = m_\bullet mm \cdots mba_0a_1a_2 \cdots a_k$$

with $\quad 0 \leqslant b < m \quad$ and $\quad 0 \leqslant a_j < \beta$

we have that $\quad {}_\bullet a_0a_1a_2 \cdots a_k < 1 \quad$ and $\quad$ we use (F)

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3. $x' \in \big[0, m\beta/(\beta - 1)\big)$

We have:

$$X^m(\gamma) = \big\{x \in \mathbb{Z}[\gamma] : x' \in \Omega\big\} \quad \text{where} \quad \Omega = \Big[0, \frac{m}{1-\gamma'}\Big)$$

Cut-and-project sets

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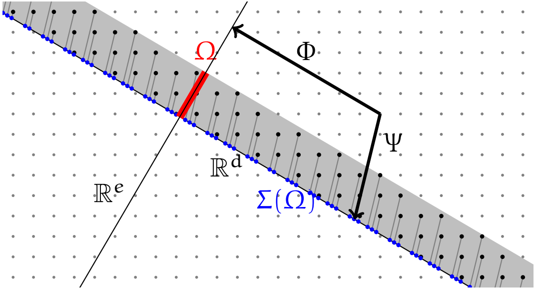
Cut-and-project sets:

- lattice $\mathbb{Z}^{d+e}$
- linear map Ψ : lattice $\to \mathbb{R}^d$ of full rank d, and injective.
- linear map Φ : lattice $\to \mathbb{R}^e$ such that $\Phi(\mathbb{Z}^{d+e})$ is dense.
- cut-and-project set $\Sigma(\Omega) := \{\Psi(v) : v \in \mathbb{Z}^{d+e}, \Phi(v) \in \Omega\}$



Cut-and-project sets

$$X^m(\gamma) = \{x \in \mathbb{Z}[\gamma] : x' \in \Omega\} \quad \text{where} \quad \Omega = \left[0, \frac{m}{1-\gamma'}\right)$$

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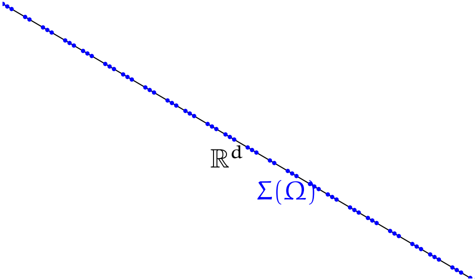


Cut-and-project sets

Properties of cut-and-project sets:

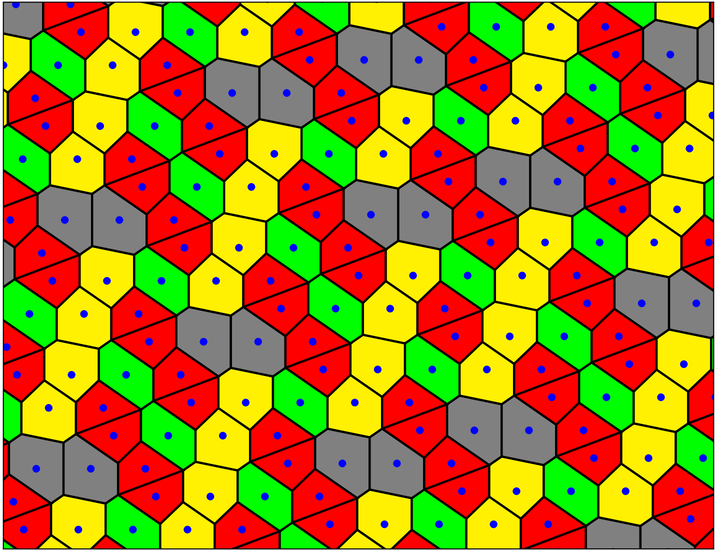
- Delone sets (uniformly discrete, relatively dense)
- (almost) repetitive
- finite local complexity (and patches are easily enumerable)
- patches change just a little when Ω changes

$$\Omega = \big[0, m/(1-\gamma')\big) \qquad m \in \mathbb{N}$$

- in our case self-similar: $\gamma\Sigma(\Omega) = \Sigma(\gamma'\Omega)$

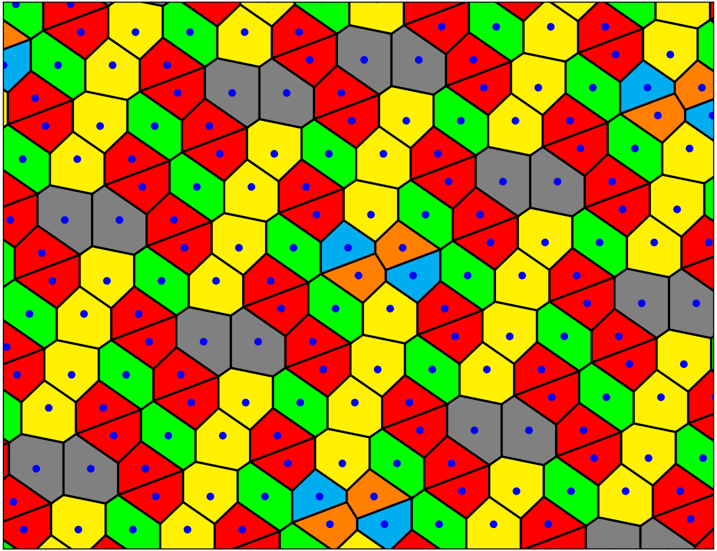
Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, C)$, $C = 2/(1 - \gamma')$
- the set $X^2(\gamma)$



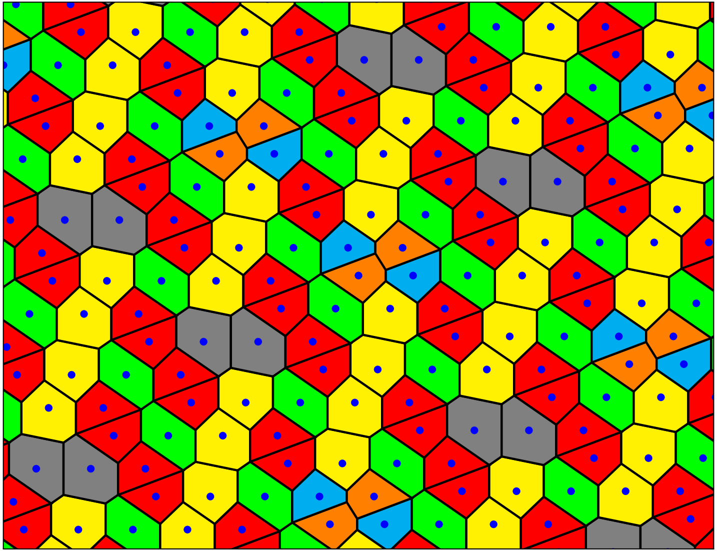
Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, 1.015C)$



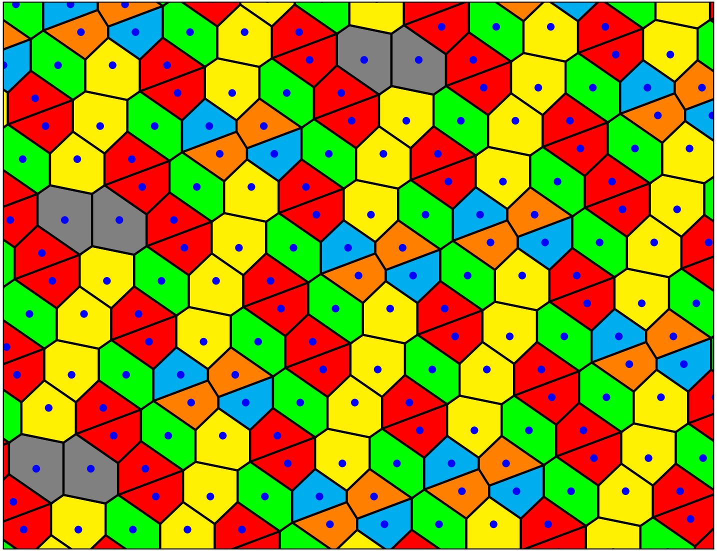
Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, 1.025C)$



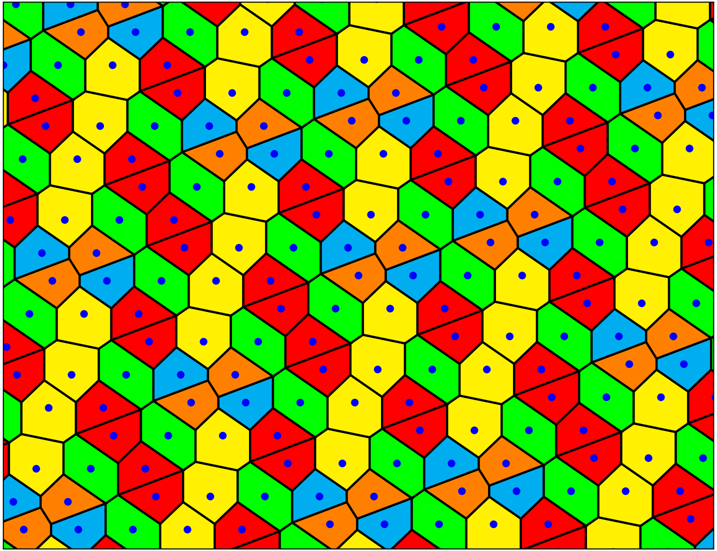
Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, 1.05C)$



Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, 1.065C)$



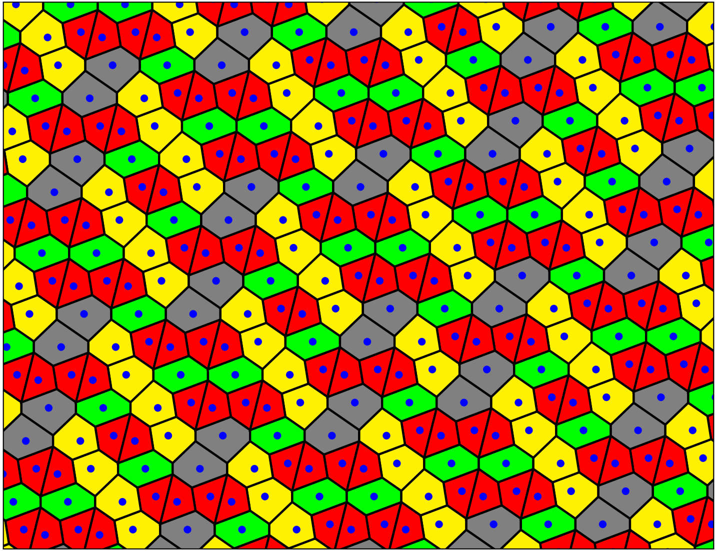
Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$

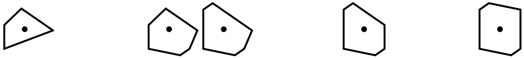
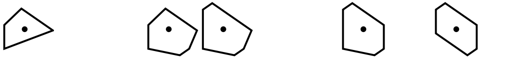
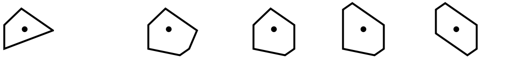
etc.

Example

- $\gamma^3 + \gamma^2 + \gamma - 1 = 0$
- $\gamma \approx -0.771 + 1.115i$
- $\Omega = [0, (1/\gamma')C)$



All tiles

Interval for c	The palette of $\Sigma(\Omega)$, where $\Omega = [0, c)$
β^2	
$(\beta^2, 2\beta)$	
$(2\beta, \beta + 2)$	
$(\beta + 2, \beta^2 + 1)$	
$(\beta^2 + 1, 2\beta + 1)$	
$(2\beta + 1, \beta^2 + \beta)$	
$(\beta^2 + \beta, \beta^2 + 2)$	
$(\beta^2 + 2, 2\beta + 2)$	
$(2\beta + 2, \beta^3)$	

Values of L_m and ℓ_m

Theorem

Let γ be the complex Tribonacci constant, $\gamma^3 = -\gamma^2 - \gamma + 1$. and $m \in \mathbb{N}$. Let $k \in \mathbb{Z}$ be the maximal integer such that $m \geqslant (1-\gamma')(\gamma')^{-k}$. Then

$$\ell_m(\gamma) = |\gamma|^{-k} \quad \textit{and} \quad L_m(\gamma) = 2\sqrt{\frac{1-(\gamma')^2}{3-(\gamma')^2}}|\gamma|^{3-k}.$$

- Question: When is $\ell_m(\gamma)$ a unit?
- Conjectured answer: For our case (CCPU and $1/\gamma'$ has (F)) always.

Conclusions

- ✚ We know that $m + 1 < |\gamma|^2$ imples $X^m(\gamma)$ not relatively dense
- ✚ We determine $\ell^m(\gamma)$ and $L^m(\gamma)$ simultaneously for all m for a large class of cubic numbers
- ℹ Tools:
  - ℹ Cut-and-project sets
  - ℹ Voronoi tiles
  - ℹ Beta-numeration
- ? What to do when Property (F) is missing?
  - ✚ forget Rényi expansions and use general representations
- ? Are cubic units the only cut-and-project case?
  - ✚ Some $\gamma = i\sqrt{\beta}$ with β real Pisot fit into the scheme as well
  - ? What about non-units? (p-adic places?)