Balancedness of Arnoux-Rauzy and Brun Words

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Outline

1. Introduction: S-adic words and continued fractions
2. Results
3. Proof ideas

$\mathcal{S}$-adic systems

Example: **Sturmian Words**

- Two substitutions: $\phi_1 : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 12 \end{array}$ $\quad \phi_2 : \begin{array}{l} 1 \mapsto 21 \\ 2 \mapsto 2 \end{array}$

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- Prefixes of $\mathcal{S}$-adic word:

$$\begin{aligned}
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- This way we can get all (standard) Sturmian words

Arnoux-Rauzy words on d letters

- **Arnoux-Rauzy substitutions** for $d = 3$:

$$\alpha_1 : \begin{matrix} 1 \mapsto 1 \\ 2 \mapsto 12 \\ 3 \mapsto 13 \end{matrix} \qquad \alpha_2 : \begin{matrix} 1 \mapsto 21 \\ 2 \mapsto 2 \\ 3 \mapsto 23 \end{matrix} \qquad \alpha_3 : \begin{matrix} 1 \mapsto 31 \\ 2 \mapsto 32 \\ 3 \mapsto 3 \end{matrix}$$

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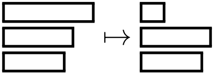
- Letter frequencies form a set of measure zero [Arnoux, Starosta]

Multi-dimensional continued fractions — Brun algorithm

Brun algorithm: for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}_+^3$

$$B_{12} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{for } x_1 > x_2 > x_3$$



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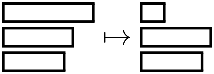

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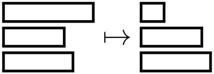

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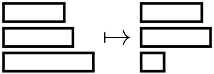

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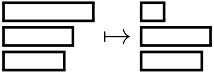

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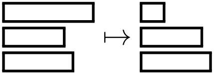

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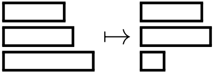

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- Forbidden sequences: $B_{12}B_{13}, \ B_{12}B_{32}, \ \ldots$
- Works for a.e. $(x_1, x_2, x_3) \in \mathbb{R}^3_+$

Multi-dimensional continued fractions — Brun algorithm

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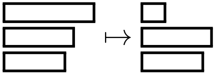

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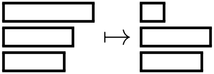

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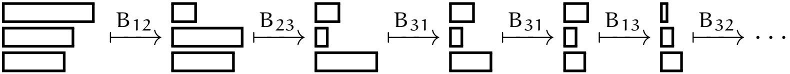
- 

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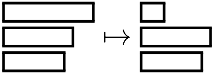

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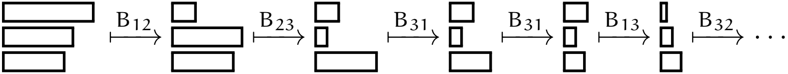
- 

S-adic: β_{12} β_{23} β_{31} β_{31} β_{13} β_{32} $\cdots$

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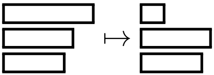

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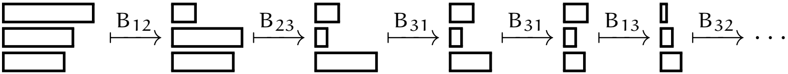
- 

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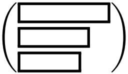
- Letter frequencies: $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \sim$ 

Balance

A word ω is C**-balanced** if

$$-C \leqslant |u|_i - |v|_i \leqslant C$$

for all pairs of factors u, v of the same length and for all letters i

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- **Finite balances** if ω is C-balanced for some C
- **Inifinite balances** otherwise

Main results

Theorem

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Theorem (Cassaigne, Ferenczi, Zamboni)

There exist Arnoux-Rauzy words that are not finitely balanced

Bounded partial quotients: Brun algorithm

$$\beta_{i_1 j_1} \beta_{i_2 j_2} \dots \beta_{i_h j_h}$$

Proposition (Avila, Delecroix)

If $\{i_1, i_2, \dots, i_{n+h-1}\} = \{1, 2, 3\}$, *then* $\beta_{i_1 j_1} \beta_{i_2 j_2} \dots \beta_{i_h j_h}$ *is Pisot and primitive*

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Proposition

Let

$$\beta_{i_1 j_1} \beta_{i_2 j_2} \beta_{i_3 j_3} \beta_{i_4 j_4} \beta_{i_5 j_5} \beta_{i_6 j_6} \beta_{i_7 j_7} \beta_{i_8 j_8} \cdots$$

have partial quotients bounded by h*. Then the resulting Brun word is* $(4h + 7)$*-balanced.*

Bounded partial quotients: Matrix manipulations

- Given frequency vector $\mathbf{f} \in \mathbb{R}^3_+$
- Directive sequence: $\beta_{i_1 j_1} \beta_{i_2 j_2} \beta_{i_3 j_3} \beta_{i_4 j_4} \cdots$
- Matrices: $M_1, M_2, M_3, M_4, \ldots$

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Lemma

Let $N_1, N_2, \ldots$ *be matrices such that*

$$ {}^t(M_1 M_2 \cdots M_n)x = {}^t(N_1 N_2 \cdots N_n)x \quad \textit{for all } x \perp \mathbf{f}, \textit{ for all } n. $$

Then

$$ \text{Balance}(\omega) \leqslant 4 \sum_{n=0}^{\infty} \left\| {}^t(N_1 N_2 \cdots N_n) \right\|_\infty $$

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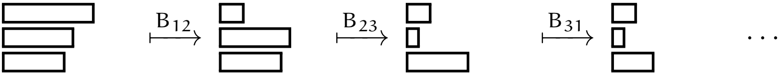

	β_{12}	β_{23}	β_{31}	$\cdots$
$M_1 \cdots M_n$	$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$	$\begin{pmatrix}2&1&1\\1&1&1\\1&0&1\end{pmatrix}$ (grows exponentially)	$\cdots$
$N_1 \cdots N_n$	$\begin{pmatrix}0&0.23&-0.67\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-0.07&0&-0.64\\-0.3&0&0.13\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-0.48&0.1&0\\-0.22&-0.02&0\\0.65&-0.15&0\end{pmatrix}$ (decays as ϕ^{-n})	$\cdots$

Bounded partial quotients: Matrix manipulations

Proof:

$$\text{Balance}(\omega) \leqslant 4\,\text{Discrepancy}(\omega) \leqslant 4 \sum_{n=0}^{\infty} \left\| {}^t(N_1 N_2 \cdots N_n) \right\|_{\infty}$$

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- **Dumont-Thomas representation:** For every prefix of ω:

$$p = \phi_1\phi_2 \ldots \phi_n(u_n) \cdot \phi_1\phi_2 \ldots \phi_{n-1}(u_{n-1}) \cdot \ \cdots \ \cdot \phi_1(u_1) \cdot u_0$$

- Brun/AR words: $|u_i| \leqslant 1$

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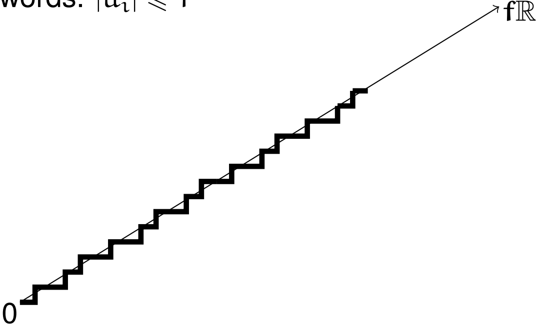

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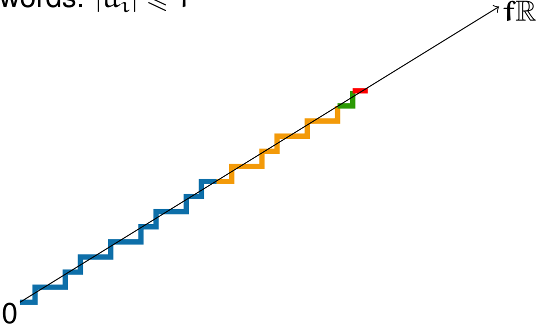

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- Lyapunov exponents $\theta_1, \dots, \theta_d$: "logarithms of eigenvalues"
- [Avila, Delecroix] $\implies \theta_1 > 0 > \theta_2$ for a.e. f
  (Oseledec's theorem – ergodic theory)

Bounded partial quotients $\to$ 'Almost all'

- Direction $f \in \mathbb{R}^3_+ \to$ cocycle $(M_1 M_2 \dots M_n)_{n\in\mathbb{N}}$
- Bounded partial quotients by $h \implies (M_1 \dots M_n)_{n\geqslant h}$ is Pisot

- Lyapunov exponents $\theta_1, \dots, \theta_d$: "logarithms of eigenvalues"
- [Avila, Delecroix] $\implies \theta_1 > 0 > \theta_2$ for a.e. f
  (Oseledec's theorem – ergodic theory)

- We have $\left\|{}^t(M_1 \dots M_n)v\right\| = \mathcal{O}\big(\exp(\theta_2 n)\big)$ for a.e. $v \perp f$

$$\mathrm{Balance}(\omega) \leqslant 4 \sum_{n=0}^{\infty} \left\|{}^t(N_1 N_2 \cdots N_n)\right\|_\infty$$

Imbalances in Brun words: Balance$(\omega) > C$

- $\Delta(u, v) = \Psi(u) - \Psi(v)$
- $\Delta\big(\phi(u), \phi(v)\big) = M_\phi \Delta(u, v)$

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- $\pm \begin{pmatrix} C \\ 1 \\ C+1 \end{pmatrix} \xleftarrow{\cdots} \cdots \xleftarrow{\cdots} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \xleftarrow{\cdots} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \xleftarrow{\cdots} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Imbalances in Brun words: $\text{Balance}(\omega) = +\infty$

Lemma

There exists a function F *such that:*
Let $\phi = \beta_1 \beta_2 \cdots \beta_n$. *Then*

$$\text{Balance}(\omega) > F(n, C) \quad \Longrightarrow \quad \text{Balance}\big(\phi(\omega)\big) > C.$$

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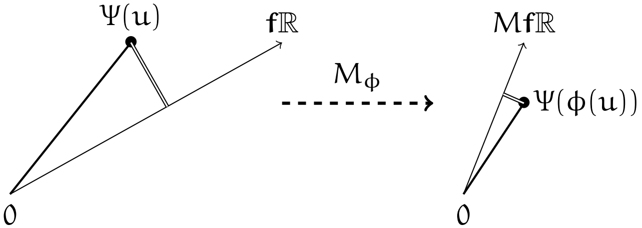

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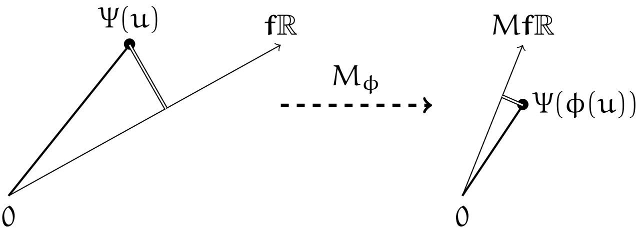

$$\underbrace{\beta_1\beta_2\cdots\beta_{n_1}}_{\text{Balance}>C_1}\ \underbrace{\beta_{n_1+1}\beta_{n_1+2}\cdots\beta_{n_2}}_{\text{Balance}>F(n_1,C_2)}\ \underbrace{\beta_{n_2+1}\beta_{n_2+2}\cdots\beta_{n_3}}_{\text{Balance}>F(n_2,C_3)}\ \cdots$$

Conclusions

- + Almost all Brun and AR words are finitely balanced.
- + But not all of them.
- i Tools:
  - i Dumont-Thomas representation
  - i Ergodic theory
  - i Linear algebra
- ? Can we characterize them?
- ? Do Brun words have a nice combinatorial description?
- ? What about other continued fraction algorithms?
  - ? Jacobi-Perron (∞ many substitutions)
  - ? Poincaré / Fully subtractive
- ? Algorithms to precisely compute Balance(ω)
- ? …

Most notable References

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