Möbius numeration systems with discrete groups

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Outline

1. Möbius numeration systems
2. Hyperbolic geometry
3. Fuchsian groups
4. Results

Motivation – Möbius number systems

- Möbius transformation: $M_{\mathbf{A}} : z \mapsto \frac{az+b}{cz+d}$ with $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ and $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) > 0$
  - Möbius transformations form a group isometric to $\mathrm{PGL}^{+}(2,\mathbb{R})$
- Möbius number system: given by $F_a, a \in \mathcal{A}$ and $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$
  - $u = u_1 u_2 \cdots u_n \quad \to \quad F_u := F_{u_1} F_{u_2} \cdots F_{u_n}$.
  - infinite word $\mathbf{u}$ represents $x \in \overline{\mathbb{R}}$ if

$$\lim_{n\to\infty} F_{\mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_n}(i) = x$$

  - condition 1: every $\mathbf{u} \in \Sigma$ is a representation
  - condition 2: every $x \in \overline{\mathbb{R}}$ has a representation
  - F_Σ is a subset of the group $\langle F_a,\ a \in \mathcal{A}\rangle$

Motivation – Möbius number systems

Example (Binary representations)

Let

$$F_\sharp : z \mapsto 2z, \quad F_0 : z \mapsto z/2, \quad F_1 : z \mapsto (z+1)/2, \quad F_{\overline{1}} : z \mapsto (z-1)/2.$$

Let Σ be set of words of the form

$$\{1,0,\overline{1}\}^\omega, \qquad \sharp^n\{1,\overline{1}\}\{1,0,\overline{1}\}^\omega, \qquad \sharp^\omega.$$

Then

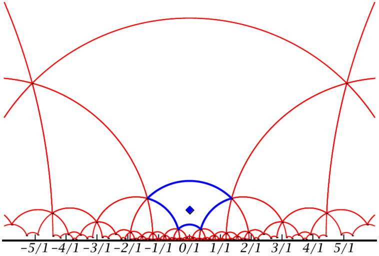
$$\sharp^n a_1 a_2 a_3 \cdots \quad \text{represents} \quad 2^n \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \qquad \sharp^\omega \quad \text{represents} \quad \infty.$$

All $x \in \overline{\mathbb{R}}$ have a representation $\implies$ (F, Σ) is a Möbius number system.

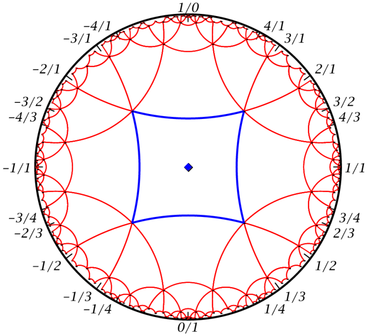
Hyperbolic plane

- Möbius transformations are isometries of the hyperbolic plane

upper half-plane $\mathbb{U}$

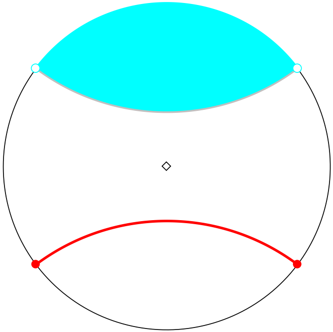


unit disc $\mathbb{D}$



Transformation properties

- Isometric circle $I(M) := \{z | M^{\bullet}(z) = 1\}$
- Expansion area $V(M) := \{z | (M^{-1})^{\bullet}(z) > 1\}$



- Example: $T : z \mapsto 4z$
- $I(M)$ $V(M)$

Fuchsian groups and Möbius number systems

- A group G of MTs is Fuchsian, if it is discrete
- A fundamental domain of G:
  such $P \subset \mathbb{U}$ that its G-images tesselate $\mathbb{U}$

Example

Group generators
$M_0(z) = (2 + 1/\sqrt{3})z$,
$M_1(z) = \frac{z\sqrt{3}+1}{z+\sqrt{3}}$

+ bounded fundamental domain
+ many group identities
− irrational

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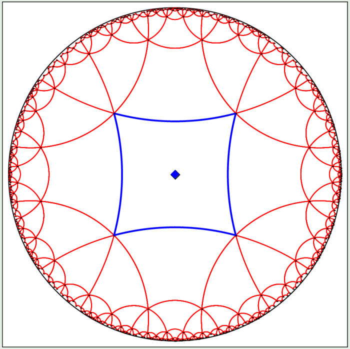

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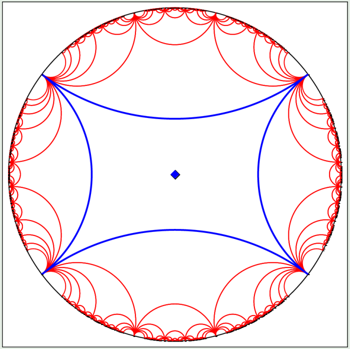
Example

Group generators
$M_0(z) = z/4$,
$M_1(z) = \frac{5z+4}{4z+5}$

− unbounded fundamental domain

− only trivial group identities

+ rational



Our interest: Rational groups

Question

Does a rational Fuchsian groups with a bounded fundamental domain exist?

Conjecture (5.1)

There is no rational Fuchsian group with a bounded fundamental domain.

Theorem (5.3)

A rational Fuchsian group contains only elements of orders $1, 2, 3, 4, 6, \infty$.

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More results — Fuchsian groups

Theorem (4.12.)

Let $G = \langle M_1, \ldots, M_k \rangle$ be a group of Möbius transformations such that none of M_j fixes i and the regions

$$V(M_1), \ldots, V(M_k),$$
$$V(M_1^{-1}), \ldots, V(M_k^{-1})$$

are pairwise disjoint. Then G is a Fuchsian group.

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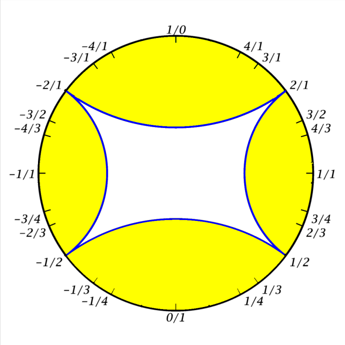

More results — generalized Ford domains

Theorem (L. R. Ford, 1925)

Let G be a Fuchsian group such that only $\mathrm{Id} \in G$ fixes i. Then the set

$$\mathbb{U} \setminus \bigcup_{\substack{M \in G \\ M \neq \mathrm{Id}}} V(M)$$

is a fundamental domain of G.

Theorem (4.9.)

Let G be a Fuchsian group with exactly r elements fixing i. Then the set

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comprises exactly r copies of a fundamental domain of G.

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Example (Modular group)

- Transformations $z \mapsto \frac{az+b}{cz+d}$
- Restrictions $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$
- The elements $z \mapsto z$ and $z \mapsto -1/z$ fix the point i

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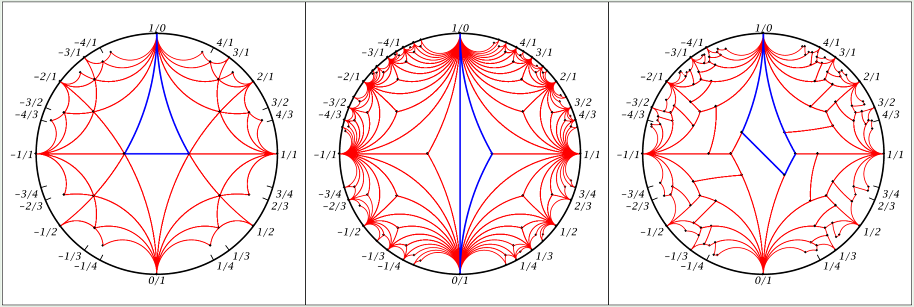
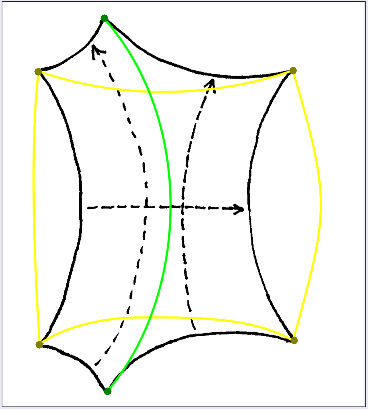


More results — Angles of rotation

- Let $M \neq \mathrm{Id}$ have a fixed point s inside $\mathbb{U}$
- Then M is a hyperbolic rotation around the point s by angle φ_M
- These M are called elliptic transformations

Theorem (4.18., explained by example)

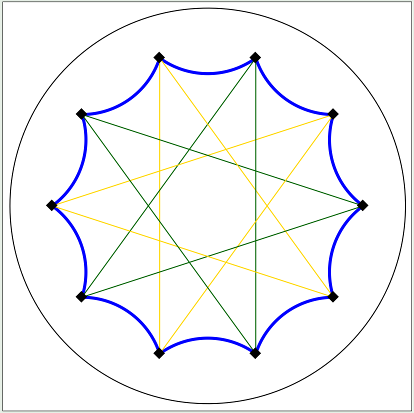
- *The theorem discuss the existence of elliptic transformations in G.*
- *The angle of rotation φ_M is sum of the angles at (some) vertices of the domain.*
- *M is identity $\iff$ $\varphi_M \in 2\pi\mathbb{Z}$.*



More results — Angles of rotation

Example

- Transformations
  $M_0 : z \mapsto \frac{\left(1+\sqrt[4]{5}\right)^2}{\left(1-\sqrt[4]{5}\right)^2} z, \ldots$
- The angles at vertices are $2\pi/5$
- Therefore:
  $F_0 F_1^{-1} F_2 F_3^{-1} F_4 = \mathrm{Id}$ (green)
  $F_0 F_4^{-1} F_3 F_2^{-1} F_1 = \mathrm{Id}$ (yellow)



Conclusions

1. We conjecture that no rational Fuchsian group with bounded fundamental domain exist.
2. We prove several statements concerning Fuchsian groups:
   - discreteness of a large family of groups;
   - shape of a fundamental domain of a special kind;
   - existence of elliptic transformations in a group.

Most notable references

1. Alan F. Beardon. *The geometry of discrete groups.* 1983.
2. Lester R. Ford. *The fundamental region for a Fuchsian group.* 1925.
3. Svetlana Katok. *Fuchsian groups.* 1992.
4. Petr Kůrka. *A symbolic representation of the real Möbius group.* 2008.
5. John M. H. Olmsted. *Discussions and Notes: Rational Values of Trigonometric Functions.* 1945.
6. A. Rényi. *Representations for real numbers and their ergodic properties.* 1957.

Compact fundamental domains

U jiných klíčových tvrzení pak čtenáři nezbyde, aby si je zformuloval sám — například, že ***kompaktnost fundamentální domény je vlastností grupy a nezávisí na volbě fundamentální domény***

- We do not claim or discuss such proposition, because it is not necessary
- We discuss *existence* of a compact fundamental domain
- Existence of non-compact f.d. is not necessarily relevant.

Theorem 4.12.

Theorem

Let $G = \langle M_1, \ldots, M_k \rangle$ be a ~~Fuchsian~~ group such that none of M_j fixes i and the regions $V(M_1), \ldots, V(M_k)$, $V(M_1^{-1}), \ldots, V(M_k^{-1})$ are pairwise disjoint. Then G is a Fuchsian group.

Proof.

- [Beardon, Theorem 8.4.1]: G is Fuchsian $\iff$ the fixed points of elliptic elements do not accumulate at identity.
- This is true since if they accumulated at identity, P and $M(P)$ would overlap for some elliptic M (and we know that $P \cap M(P) = \emptyset$ for all $M \neq \mathrm{Id}$).

□

Example 2.4

Example (In thesis)

- Subshift: $\Sigma = \{\sharp^{\mathbb{N}}, 0^{\mathbb{N}}\} \cup (\sharp^* \cup 0^*)\big(\{-b, \dots, -1\}\{-b, \dots, 0\}^{\mathbb{N}} \cup \{1, \dots, b\}\{0, \dots, b\}^{\mathbb{N}}\big)$
- Forbidden strings: $X = \{\sharp 0\} \cup \left\{a\sharp \;\middle|\; a \in \{-b, \dots, b\}\right\} \cup \left\{aa' \;\middle|\; a, a' \in \{-b, \dots, b\}, a \cdot a' < 0\right\}$
- X allows $a0^+(-a)$, Σ does not (it is not SFT)

Example (Correct)

- Subshift: distinguish $0 \in \{0, \dots, b\}$ and $-0 \in \{-b, \dots, -0\}$
- Forbidden strings: $X = \{\sharp 0, \sharp(-0)\} \cup \left\{a\sharp \;\middle|\; a \in \{-b, \dots, b\}\right\} \cup \left\{aa', a'a \;\middle|\; a \in \{0, \dots, b\}, a' \in \{-b, \dots, -0\}\right\}$
- X is finite and Σ is SFT.