On the negative base greedy and lazy representations

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Numeration systems

Numeration system is a system of representations of (some) real numbers.

Real positional numeration systems

Definition

Given: $\alpha \in \mathbb{R}$ with $|\alpha| > 1$, finite alphabet $\mathcal{A} \in \mathbb{R}$.

String $\bullet a_1 a_2 a_3 \ldots \in \mathcal{A}^{\mathbb{N}}$ is $(\alpha, \mathcal{A})$-**representation** of $x \in \mathbb{R}$, if

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \frac{a_3}{\alpha^3} + \cdots$$

Which numbers have a representation

Let $\mathcal{I}_{\alpha,\mathcal{A}} = \{x \mid x \text{ has an } (\alpha,\mathcal{A})\text{-representation}\}$.

- Case $\alpha = \beta > 1$: (M. Pedicini, 2005) Necessary and sufficient condition for $\mathcal{I}$ to be an interval.

  It suffices: $\mathcal{A} = \{0, \dots, \lfloor\beta\rfloor\}$

  $$\mathcal{I} = \left[0, \frac{\lfloor\beta\rfloor}{\beta - 1}\right]$$

- Case $\alpha = -\beta < -1$:

  $$\mathcal{I} = \left[-\frac{\beta\lfloor\beta\rfloor}{\beta^2 - 1}, \frac{\lfloor\beta\rfloor}{\beta^2 - 1}\right]$$

How to get a representation

Approach ONE: Algorithm

- Give: function $D : \mathcal{I} \mapsto \mathcal{A}$.
- Restriction: $T(x) := \alpha x - D(x) \in \mathcal{I}$ for all $x \in \mathcal{I}$.
- Representation:

$$x = \frac{D(x)}{\alpha} + \frac{D(T(x))}{\alpha^2} + \frac{D(T^2(x))}{\alpha^3} + \cdots$$

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Example (Binary representations)

Let

$$\alpha := 2, \qquad \mathcal{A} := \{0,1\}, \qquad D(x) := \lfloor 2x \rfloor$$

Then

$$T(x) = 2x - \lfloor 2x \rfloor = \{2x\} \in [0,1) \quad \text{and} \quad [0,1) \subseteq \mathcal{I}.$$

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Example (Rényi, 1957)

Let

$$\alpha := \phi = \frac{1+\sqrt{5}}{2}, \qquad D(x) = \lfloor \phi x \rfloor, \qquad A = \{0, 1\}$$

Then

$$T(x) = \phi x - \lfloor \phi x \rfloor = \{\phi x\} \in [0, 1) \quad \text{and} \quad [0, 1) \subseteq \mathcal{I} = [0, \phi].$$

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Example (Ito, Sadahiro, 2009)

Let

$$\alpha := -\phi, \qquad \mathcal{A} = \{0,1\}, \qquad D(x) = \lfloor -\phi x + \tfrac{1}{\phi} \rfloor$$

Then

$$T(x) := \{-\phi x + \tfrac{1}{\phi}\} - \tfrac{1}{\phi} \in [-\tfrac{1}{\phi}, \tfrac{1}{\phi^2}) \quad \text{and} \quad [-\tfrac{1}{\phi}, \tfrac{1}{\phi^2}) \subseteq \mathcal{I} = [-\tfrac{1}{\phi}, \tfrac{1}{\phi^2}).$$

How to get a representation

Approach TWO: Criteria

- For $x \in \mathcal{I}$ consider all $\bullet a_1 a_2 a_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ such that $x = \sum a_k \alpha^{-k}$
- This (in general) allows multiple representations of x
- Give a criterion saying which one to choose

Idea: extremal representations

Extremal representations

Definition (Lexicographical ordering)

Let $\mathbf{a} = \bullet a_1 a_2 a_3 \cdots$ and $\mathbf{b} = \bullet b_1 b_2 b_3 \cdots$ be representations.
Then $\mathbf{a} \prec_{\mathrm{lex}} \mathbf{b}$ if

$$a_k < b_k \quad \text{for} \quad k = \min\{i \geq 1 | a_i \neq b_i\}.$$

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Definition (Alternate ordering)

Let $\mathbf{a} = \bullet a_1 a_2 a_3 \ldots$ and $\mathbf{b} = \bullet b_1 b_2 b_3 \cdots$ be representations.
Then $\mathbf{a} \prec_{\text{alt}} \mathbf{b}$ if

$$(-1)^k a_k < (-1)^k b_k \quad \text{for} \quad k = \min\{i \geq 1 | a_i \neq b_i\}.$$

Extremal representations

Definition (Lazy and greedy representations)

Let $b > +1$ be a positive base. The maximal representation with respect to the lexicographical order is called the **greedy representation**, the minimal one is the **lazy represetation**.

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Proposition (K. Dajani, C. Kalle, 2010)

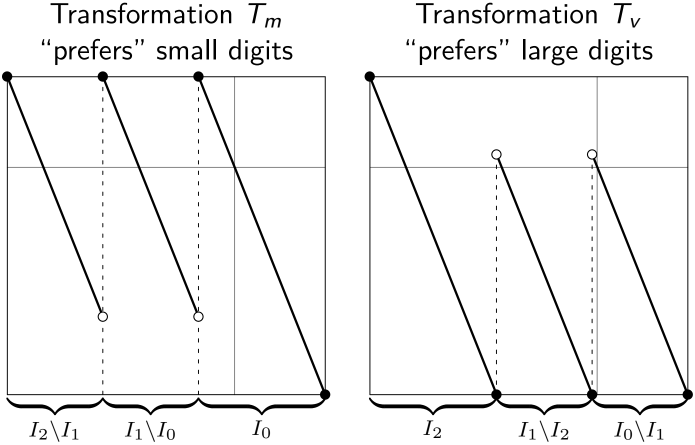
There is no transformation (approach ONE) generating lazy and greedy representations in negative base.

How to obtain extremal representations

We suppose: $\alpha = -\beta \notin \mathbb{Z}$, $\qquad \mathcal{A} = \{0, \ldots, \lfloor \beta \rfloor\}$

Stated in the means of algorithm by (K. Dajani, C. Kalle, 2010)

How to obtain extremal representations I



Example for $2 < \beta < 3$

How to obtain extremal representations II

Greedy "transformation":

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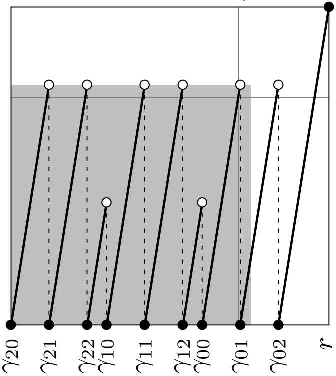

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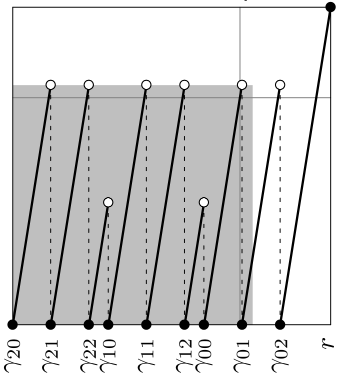
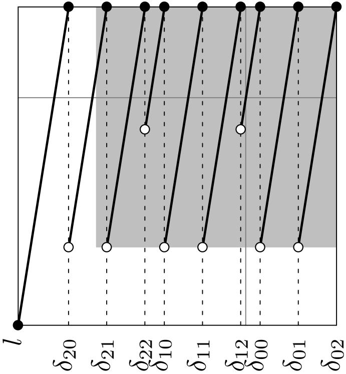


Lazy "transformation":

- use T_v on odd positions
- use T_m on even positions



Alternation of transformations

- Symmetry between greedy and lazy case $\implies$ we will focus on greedy
- Define: $T_G(x) = T_v T_m(x)$ and $D_G = \beta^2 x - T_G(x)$
- D_G uses digits from the alphabet $\mathcal{B} = \{-\beta b + a \mid a, b \in \mathcal{A}\}$
- T_G is a well-defined β^2-transformation with a positive base

Step-by-step how-to for greedy $(-\beta)$-representations

Three steps:

1. Represent x using T_G, D_G in alphabet $\mathcal{B} = -\beta\mathcal{A} + \mathcal{A}$
2. Substitution $\mathcal{B} \mapsto \mathcal{A}$: $\quad -\beta b + a \mapsto ba$
3. The result is the greedy $(-\beta, \mathcal{A})$-representation or x

- Main idea:

$$ba \prec_{\text{alt}} dc \quad \Longleftrightarrow \quad -b\beta + a < -d\beta + c \qquad \text{for } a, b \in \mathcal{A}$$

Main advantage: Allows to use general knowledge of radix transformations

Admissibility conditions

- Using result from (C. Kalle, W. Steiner, 2012)

Theorem

Let $X_1X_2X_3\cdots \in \mathcal{B}^{\mathbb{N}}$. *Then there exists* $x \in [l, l+1)$ *such that* $d_G(x) = X_1X_2X_3\cdots$

if and only if

for every $k \geq 1$ *we have* $X_k \leq D_G^*(l+1)$ *and*

$$X_{k+1}X_{k+2}X_{k+3}\cdots \prec_{\mathrm{lex}} \begin{cases} d_G^*(T_G^*(l+1)) & \text{if } X_k = D_G^*(l+1), \\ d_G^*(l+\{\beta\}) & \text{if } X_k = -b\beta + \lfloor\beta\rfloor < D_G^*(l+1), \\ \text{no condition} & \text{otherwise,} \end{cases}$$

where d_G^* *and* T_G^* *are left-continuous modifications of* d_G *and* T_G.

Forbidden strings

For some β, we can get forbidden strings.

Corollary

String $\bullet a_1 a_2 a_3 \cdots$ over $\mathcal{A} = \{0, 1\}$ is a greedy representation in base $-\phi$ if and only if

1. *none of $1^{2k}0$ nor $0^{2k-1}1$ is its prefix;*
2. *none of 0^ω nor 1^ω is its infinite suffix;*
3. *none of $10^{2k}1$ nor $01^{2k}0$ is its factor.*

Unique representations

- x has a unique representation $\iff d_G(x) = d_L(x)$

Theorem (K. Dajani, C. Kalle, 2010)

The numbers with unique $(-\beta)$-representation are of Lebesgue measure zero, for all $\beta > 1$. There are only two such points for all $\beta \leq \phi$.

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Proposition

Let $\mu = 1.839\cdots$ be the 'Tribonacci constant', root of $x^3 - x^2 - x - 1$. Then all strings over the pairs of digits $\{01, 10\}$ are admissible as both greedy and lazy $(-\mu)$-represenations.

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Theorem

Let $\beta > 1 + \sqrt{3} = 2.732$. Then there are uncountably many numbers with unique $(-\beta)$-representations over the alphabet $\{0, \ldots, \lfloor\beta\rfloor\}$.

Optimal representations

- Another "extremal" representation

Definition (C. Dajani, M. de Vries, V. Komornik, P. Loreti, 2011)

String $\bullet a_1 a_2 a_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ is optimal $(\alpha, \mathcal{A})$-representation if

$$\left| x - \sum_{k=1}^{n} \frac{a_k}{\alpha^k} \right| \leq \left| x - \sum_{k=1}^{n} \frac{b_k}{\alpha^k} \right|$$

for each $n \geq 1$ and each $\bullet b_1 b_2 b_3 \cdots \in \mathcal{A}^{\mathbb{N}}$.

- D+dV+K+L study for $\alpha > 1$
- Properties for $\alpha < -1$ and $\alpha \in \mathbb{C}$?

Conclusions

- Stating the algorithm for greedy and lazy in the means of transformations
- Lexicographic admissibility conditions
- Results on unique representations

Open problems:

- Dynamical properties (ergodicity, exactness, invariant measures, ...)
- Full description of unique representations
- Optimal representations

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