Gaps in Ito-Sadahiro transformation
... and more ...

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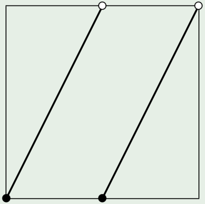
Dynamical system

We need:

- topological space X, e.g. an interval
- transformation on X, a map $X \mapsto X$
- (T, X) is dynamical system if
  1. either T is continuous on X
  2. or there exist a measure μ on X that is T-invariant

Example (Doubling map)



- space $X = [0, 1)$
- transformation $T(x) = 2x \bmod 1 = 2x - \lfloor 2x \rfloor$
- it is continuous (after some simple modification)

Measures

Measure on X is (for us) μ : (Borel sets in X) $\mapsto \mathbb{R}$ such that:

1. $\mu(B) \geq 0$ for all Borel $B \subseteq X$
2. $\mu(\emptyset) = 0$
3. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ for all Borel $A, B \subseteq X$

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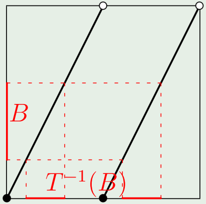


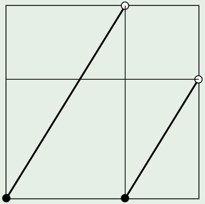
- space $X = [0, 1)$
- transformation $T(x) = 2x \bmod 1 = 2x - \lfloor 2x \rfloor$
- the invariant measure is $\mu(B) = \int_B 1 \, \mathrm{d}x$

Invariant measures

Measure μ is T-invariant if

$$\mu(B) = \mu(T^{-1}(B)) \quad \text{for all} \quad B \text{ Borel}$$

Example ($+\phi$ transformation)



- space $X = [0, 1)$
- transformation $T(x) = \phi x - \lfloor \phi x \rfloor$
- not continuous
- invariant measure is $\mu(B) = \int_B h(x)\,\mathrm{d}x$

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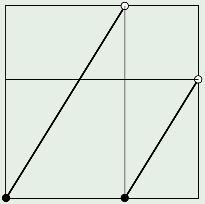

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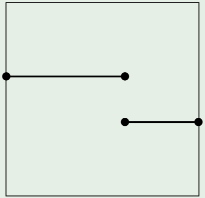
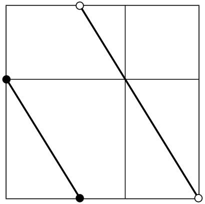


Minus-beta transformation

Ito-Sadahiro definition:

- interval $J = [\ell_\beta, r_\beta) = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$
- transformation $T(x) = -\beta x - \lfloor -\beta x - \ell_\beta \rfloor$
- digit function $D(x) = \lfloor -\beta x - \ell_\beta \rfloor \in \{0, \dots, \lfloor \beta \rfloor\}$
- Ito-Sadahiro expansion of $x \in J$ is $d(x) = 0 \bullet d_1 d_2 d_3 \cdots$ where

$$d_n = D(T^{n-1}(x)) \qquad \text{and we get} \qquad x = \frac{d_1}{(-\beta)^1} + \frac{d_2}{(-\beta)^2} + \frac{d_3}{(-\beta)^3} + \cdots$$



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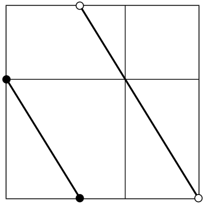

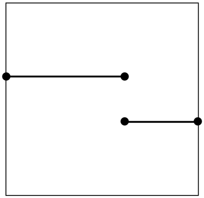

Minus-beta transformation — invariant measure

Theorem (Ito & Sadahiro, 2009)

Let $-\beta < -1$*. Define* $h : I \mapsto \mathbb{R}$ *as*

$$h(x) = \sum_{\substack{n \geq 0 \\ x \geq T^n(\ell_\beta)}} \frac{1}{(-\beta)^n}.$$

Then the measure $\mu(B) = \int_B h(x)\,\mathrm{d}x$ *is* T*-invariant measure.*

Minus-beta transformation — invariant measure

Proof.

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Q.E.D.

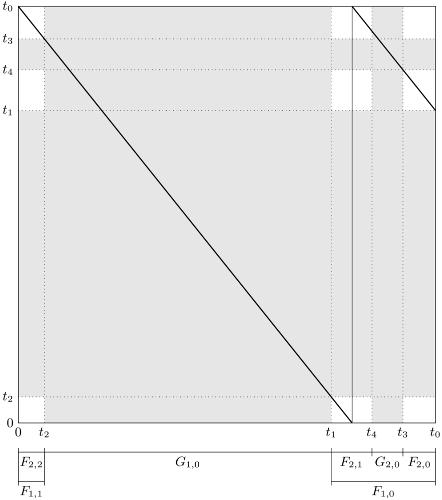
1st example of "gaps"

Example

$\beta = 1.1347241384$, root of $X^6 - X - 1 = 0$, let $s_i = T^i(\ell)$. Then $s_i + 3 = s_i$ for all $i \geq 5$

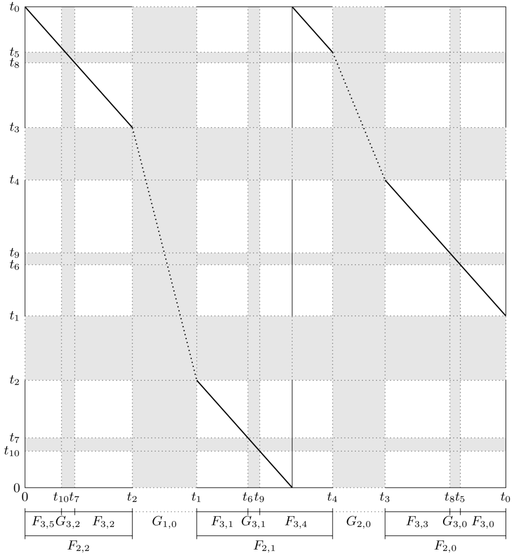
	$s_0 \sim$	$s_5 \sim$	$s_3 \sim$	$s_4 \sim$	$s_6 \sim$	$s_1 \sim$	$s_2 \sim$	$s_7 \sim$
1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta}$						$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^2}$							$\surd$	$\surd$
$-\frac{1}{\beta^3}$			$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^4}$				$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta^5}$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^6}$					$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta^7}$								$\surd$
$\frac{1}{\beta^8}$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$h_{-\beta}$	1	$\frac{1}{\beta^3}$	0	$\frac{1}{\beta^4}$	$\frac{1}{\beta}$	0	$\frac{1}{\beta^2}$	$\frac{1}{\beta^5}$

2nd example of "gaps", $\beta = 5/4$



(Image by W. Steiner and L. Liao)

3rd example of “gaps”, $\beta = 9/8$



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Topological properties of transformations

- locally eventually onto if for any non-empty open subset $U \subseteq X$ there exists $k \geq 0$ such that $T^k(U) = X$
- exact on a probabilistic space if $\lim_{n\to\infty} \mu(T^n(A)) = \mu(X)$ for all A with $\mu(A) > 0$
- l.e.o. $\implies$ exact

Liao, Steiner — results on gaps I

Definition

Let $\gamma_n > 1$ be a root of $X^{g_n+1} - X - 1 = 0$ with $g_n = \lfloor 2^{n+1}/3 \rfloor$.

Definition

For m, k, β, let

$$G_{m,k}(\beta) = \begin{cases} \left(T_{-\beta}^{(2^{m+2}-(-1)^m)/3+k}(\ell), T_{-\beta}^{2^{m+1}+k}(\ell)\right) & \text{if } k \text{ is even} \\ \left(T_{-\beta}^{2^{m+1}+k}(\ell), T_{-\beta}^{(2^{m+2}-(-1)^m)/3+k}(\ell)\right) & \text{if } k \text{ is odd} \end{cases}$$

Definition

For β, n, let $\mathcal{G}_n(\beta) = \left\{ G_{m,k}(\beta) \middle| 0 \le m < n, 0 \le k < \frac{2^{m+1}+(-1)^m}{3} \right\}$

Theorem

Let $\gamma_{n+1} \le \beta < \gamma_n$. Then the transformation $T_{-\beta}$ has exactly g_n gaps, they are the intervals in $\mathcal{G}_n(\beta)$

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For any $\beta > 1$ the transformation T is locally eventually onto on $[\ell, \ell+1) \setminus G(\beta)$, where $G(\beta) = \cup_{I \in \mathcal{G}_n(\beta)} I$

Theorem (Góra proved this for $\beta > \gamma_2$)

For any $\beta > 1$ the transformation T is exact with respect to its unique absolutely continuous invariant measure.

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Every Yrrap number is a Perron number.

- $\beta > 1$ is Yrrap if $d(\ell)$ is eventually periodic
- $\beta > 1$ is Perron if all its conjugates β' satisfy $|\beta'| < \beta$

Theorem

The expansion of ℓ in the base γ_n is

$$d(\ell) = \varphi^{n-1}(10^\omega), \qquad \textit{where} \qquad \varphi : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 100 \end{array}.$$

The expansion of ℓ in the base $1 < \beta \leq \gamma_n$ starts with $\varphi^n(1)$, hence

$$d(\ell) \xrightarrow{\beta \to 1} \varphi^\omega(1) = 100111001001001110011\cdots$$

Liao, Steiner — results on expansion of ℓ

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Few drops of “Wolfgangian”

- let $f_a : x \mapsto -\beta x + \alpha$
- let $f_{a_1 \cdots a_k} = f_{a_k} \circ \cdots \circ f_{a_1}$
- we have $f_{a_1 \cdots a_k}(1) = (-\beta)^k + \sum_{j=1}^{k} a_j (-\beta)^{k-j}$
- let $P_{a_1 \ldots a_k} = (-X)^k + \sum_{j=1}^{k} a_j (-X)^{k-j}$
- we have $f_{a_1 \cdots a_k}(1) = P_{a_1 \ldots a_k}(\beta)$
- $P_{a_1 \ldots a_k} = P_{b_1 \ldots b_l}$ iff $a_1 \ldots a_k = b_1 \ldots b_l$

Lemma

For every $n \geq 0$ we have

$$X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}.$$

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$$|\varphi^n(2)| = g_{n+1} + \frac{1-(-1)^n}{2} \qquad \text{and} \qquad |\varphi^n(11)| = g_{n+1} + \frac{1+(-1)^n}{2}.$$

Few drops of "Wolfgangian"

- let $f_a : x \mapsto -\beta x + \alpha$
- let $f_{a_1 \cdots a_k} = f_{a_k} \circ \cdots \circ f_{a_1}$
- we have $f_{a_1 \cdots a_k}(1) = (-\beta)^k + \sum_{j=1}^{k} a_j (-\beta)^{k-j}$
- let $P_{a_1 \ldots a_k} = (-X)^k + \sum_{j=1}^{k} a_j (-X)^{k-j}$
- we have $f_{a_1 \cdots a_k}(1) = P_{a_1 \ldots a_k}(\beta)$
- $P_{a_1 \ldots a_k} = P_{b_1 \ldots b_l}$ iff $a_1 \ldots a_k = b_1 \ldots b_l$

Lemma

For every $n \geq 0$ we have

$$X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}.$$

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More drops of “Wolfgangian”

Lemma

For every $n \geq 0$ the words $\varphi^n(2)$ and $\varphi^n(11)$ agree on the first $g_{n+1} - 1$ letters and differ on the g_{n+1}-st letter.

Proof. **Q.E.D.**

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