Greedy and lazy expansions in the negative base $-\phi$

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Numeration systems

Numeration system is a system of representations of (some) real numbers.

We consider:

- real case;
- base $\beta \in \mathbb{R}$ with $|\beta| > 1$;
- real numbers from some finite interval.

Definition

String $\mathbf{a} = \bullet a_1 a_2 a_3 \ldots$ **represents** $x \in \mathbb{R}$ in the base β, if

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \cdots$$

Terminology: representation (any string) vs. expansion (algorithm).

Numeration systems — classical approach for the base b

Give:

- an interval $\mathcal{I}$;
- a transformation $T : \mathcal{I} \mapsto \mathcal{I}$.

Compute:

- the digit function $D(x) := \beta x - T(x)$;
- the alphabet of digits $\mathcal{A} := D(\mathcal{I})$

Restriction: The alphabet is finite.

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Example (Binary representations)

Let

$$\beta := 2, \qquad \mathcal{I} := [0, 1), \qquad T(x) := \{2x\}$$

Then

$$D(x) = 2x - \{2x\} = \lfloor 2x \rfloor \quad \text{and} \quad \mathcal{A} = \{0, 1\}.$$

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Example (Rényi representations in the golden mean base)

Let

$$\beta := \phi = \frac{1+\sqrt{5}}{2}, \qquad \mathcal{I} := [0,1), \qquad T(x) := \{\phi x\}$$

Then

$$D(x) = \phi x - \{\phi x\} = \lfloor \phi x \rfloor \quad \text{and} \quad \mathcal{A} = \{0,1\}.$$

Numeration systems — classical approach for the base b

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Restriction: The alphabet is finite.

Example (Ito-Sadahiro representations in the minus golden mean base)

Let

$$\beta := -\phi, \qquad \mathcal{I} := [-\tfrac{1}{\phi}, \tfrac{1}{\phi^2}), \qquad T(x) := \{-\phi x + \tfrac{1}{\phi}\} - \tfrac{1}{\phi}$$

Then

$$D(x) = \lfloor -\phi x + \tfrac{1}{\phi} \rfloor \quad \text{and} \quad \mathcal{A} = \{0, 1\}.$$

Numeration systems — our approach

Give:

- a base β with $|\beta| > 1$;
- an alphabet $\mathcal{A} \subseteq \mathbb{R}$ (not necessarily integer);
- some condition **C** on the representations.

Compute:

- the maximal interval $\mathcal{I} = \left\{ \sum_{i=1}^{\infty} a_i \beta^{-i} \middle| a_i \in \mathcal{A} \right\}$;
- the algorithm to obtain the representation $(x)_{\beta,\mathcal{A},\mathbf{C}} = \bullet a_1 a_2 a_3 \ldots$, let us call it "iterating a transformation"

Restriction:

- condition **C** is "nice";
- the iteration of the "transformation" stays in $\mathcal{I}$.

Extremal representations

Definition (Lexicographical ordering)

Let $\mathbf{a} = \bullet a_1 a_2 a_3 \dots$ and $\mathbf{b} = \bullet b_1 b_2 b_3$ be representations. Then $\mathbf{a} \prec_{\text{lex}} \mathbf{b}$ if

$$a_k < b_k \quad \text{for} \quad k = \min\{i \geq 1 | a_i \neq b_i\}.$$

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Definition (Alternate ordering)

Let $\mathbf{a} = \bullet a_1 a_2 a_3 \ldots$ and $\mathbf{b} = \bullet b_1 b_2 b_3$ be representations. Then $\mathbf{a} \prec_{\text{alt}} \mathbf{b}$ if

$$(-1)^k a_k < (-1)^k b_k \quad \text{for} \quad k = \min\{i \geq 1 | a_i \neq b_i\}.$$

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Definition (Lazy and greedy representations)

Let $b > +1$ be a positive base. The maximal representation with respect to the lexicographical order is called the **greedy representation**, the minimal one is the **lazy represetation**.

Idea for the condition **C**: Let us take extremal representations.

Definition (Lazy and greedy representations)

Let $b > +1$ be a positive base. The maximal representation with respect to the lexicographical order is called the **greedy representation**, the minimal one is the **lazy represetation**.

Let $b < -1$ be a negative base. The maximal representation with respect to the alternate order is called the **greedy representation**, the minimal one is the **lazy represetation**.

Idea for condition **C** – positive base

Consider the base $\beta = +\phi$ and the alphabet $\mathcal{A} = \{0, 1\}$. Then $\mathcal{I} = [0, \phi]$. The number $\frac{1}{2} \in \mathcal{I}$ has representations:

Representation	condition **C**	representation of $\frac{1}{2}$
Rényi	take maximal digit in $\mathcal{I}_R = [0, 1)$	$(\frac{1}{2})_R = \bullet 0(100)^\omega = \bullet 0100100100\ldots$
Greedy	take lex. maximal	$(\frac{1}{2})_G = \bullet 0(100)^\omega = \bullet 0100100100\ldots$
Lazy	taky lex. minimal	$(\frac{1}{2})_L = \bullet 0(011)^\omega = \bullet 0011011011\ldots$

Minimal polynomial of ϕ is $x^2 = x + 1$, hence $100 \leftrightarrow 011$.

Idea for condition **C** – negative base

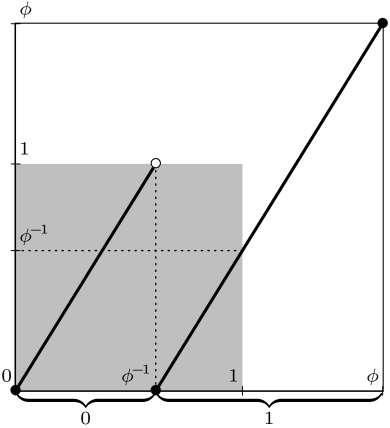
Consider the base $b = -\phi$ and the alphabet $\mathcal{A} = \{0, 1\}$. Then $\mathcal{I} = [-1, \frac{1}{\phi}]$. The number $\frac{1}{2} \in \mathcal{I}$ has representations:

Representation	condition **C**	representation of $\frac{1}{2}$
Ito-Sadahiro	take maximal digit in $\mathcal{I}_{IS} = [-\frac{1}{\phi}, \frac{1}{\phi^2})$	$(\frac{1}{2})_{IS} = \bullet(100)^\omega = \bullet 100100100\ldots$
Greedy	take alt. maximal	$(\frac{1}{2})_G = \bullet 100(111000)^\omega = \bullet 100111000\ldots$
Lazy	taky alt. minimal	$(\frac{1}{2})_L = \bullet(111000)^\omega = \bullet 111000111\ldots$

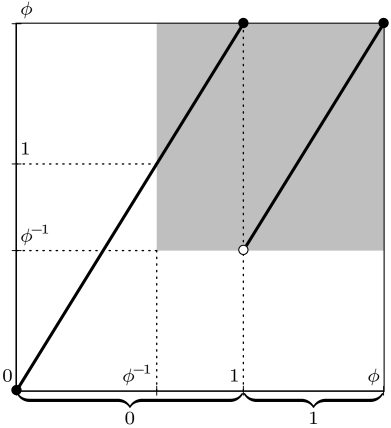
Minimal polynomial of ϕ is $x^2 - x = 1$, hence $110 \leftrightarrow 001$.

Extremal representations in a positive base

Greedy (Rényi)

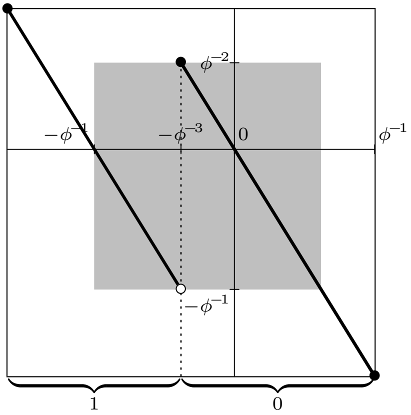


Lazy (Erdös, Joó, Komornik)

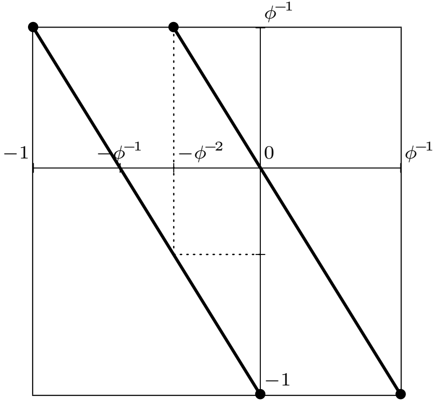


Representations in a negative base I

Ito-Sadahiro

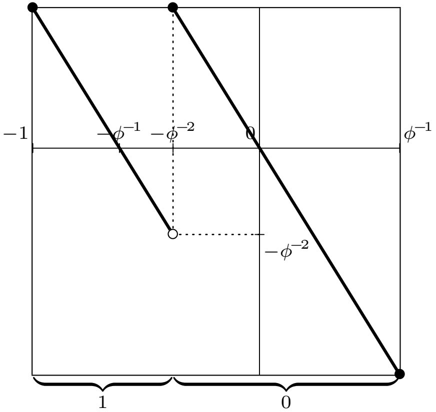


Lines $y = -\phi x - \mathcal{A}$

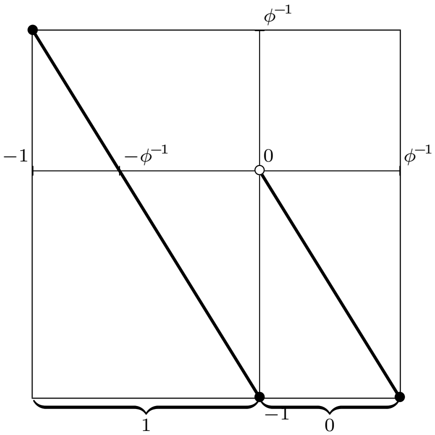


Representations in a negative base II

Transformation T_0
prefers digit 0



Transformation T_1
prefers digit 1



Representations in a negative base III

Greedy "transformation":

- use T_0 on odd positions
- use T_1 on even positions

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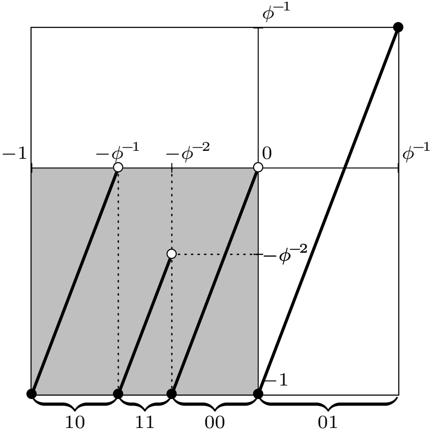

Representations in a negative base III

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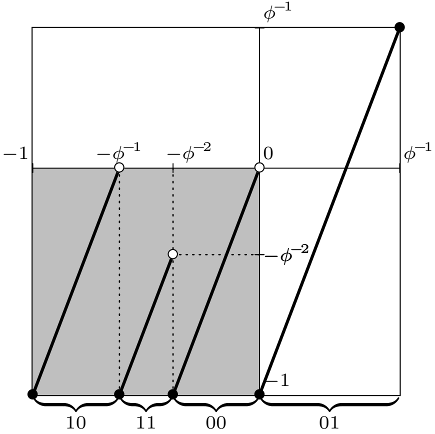
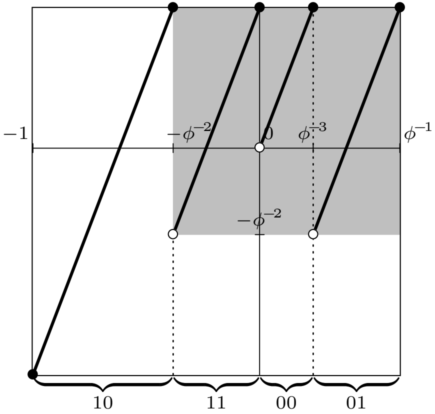


Lazy "transformation":

- use T_1 on odd positions
- use T_0 on even positions



Properties of $-\phi$ representations

Ito-Sadahiro:

- the attractor $\mathcal{I}_{IS} = [-\phi^{-1}, \phi^{-2})$ satisfies $\frac{1}{-\phi}\mathcal{I}_{IS} \subseteq \mathcal{I}_{IS}$;
- not lazy nor greedy for any $x \in \mathcal{I}_{IS}$;
- $(0)_{IS} = \bullet 0^\omega = \bullet 0000000000\cdots$

Greedy representations:

- the attractor $\mathcal{I}_G = [-1, 0)$ is whole negative and $0 \notin \mathcal{I}_G$;
- which means that the representations are not continuous in 0;
- $(0)_G = \bullet 01(10)^\omega = \bullet 0110101010\cdots$

Lazy representations:

- the attractor $\mathcal{I}_L = (-\phi^{-2}, \phi^{-1}]$ contains 0 as an interior point;
- but $\frac{1}{-\phi}\mathcal{I}_L \nsubseteq \mathcal{I}_L$;
- $(0)_L = \bullet 11(01)^\omega = \bullet 1101010101\cdots$

Connection between bases $-\phi$ and ϕ^2 I

Besides a base $-\phi$ and an alphabet $\mathcal{A} := \{0, 1\}$,
consider a base ϕ^2 and an alphabet $\mathcal{B} := \{-\phi, -\phi + 1, 0, 1\}$.

Let us define a substituion $\psi : \mathcal{B}^{\mathbb{N}} \mapsto (\mathcal{A} \times \mathcal{A})^{\mathbb{N}}$ as

$$\psi(-\phi) = 10, \quad \psi(-\phi + 1) = 11, \quad \psi(0) = 00, \quad \psi(1) = 01.$$

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We see that

$$\frac{-\phi}{\phi^2} = \frac{1}{(-\phi)^1} + \frac{0}{(-\phi)^2}, \qquad \frac{-\phi + 1}{\phi^2} = \frac{1}{(-\phi)^1} + \frac{1}{(-\phi)^2} \qquad \text{etc.}$$

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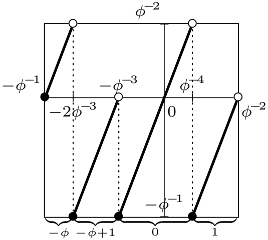
Proposition

Let $\mathbf{b} = b_1 b_2 b_3$ be a representation of $x \in \mathbb{R}$ in the base ϕ^2 with the alphabet $\mathcal{B}$. Then

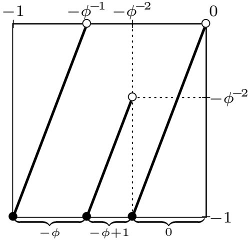
$$\psi(\mathbf{b}) = \psi(b_1)\psi(b_2)\psi(b_3)\cdots$$

is a representation of the same x in the base $-\phi$ with the alphabet $\mathcal{A}$.

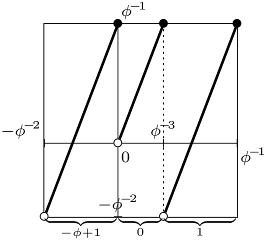
Connection between bases $-\phi$ and ϕ^2 II



Ito-Sadahiro
uses all digits from $\mathcal{B}$



Greedy
0 is a boundary point



Lazy

Conclusions and remarks

1. We show how to obtain lazy and greedy representations in negative bases.
2. The results can be generalized for any negative base $\beta < -1$.
3. The negative bases can be studied through positive bases using non-integer alphabets.
4. This is true as well for complex bases β with $|\beta| > 1$ as far as $\arg \beta \in \pi\mathbb{Q}$.

Most important references

1. HeMaPe. Greedy and lazy representations of numbers in the negative golden ratio base. Preprint 2011.
2. Shunji Ito and Taizo Sadahiro. Beta-expansions with negative bases. Integers, 9:A22, 239–259, 2009.
3. Alfréd Rényi. Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar, 8:477–493, 1957.