Möbius number systems with discrete groups

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Abstract

We study number systems generated by Möbius transformations (MT) of the hyperbolic plane $\mathbb{U} = \{z \in \mathbb{C} | \Im z \geq 0\}$. We are concerned about finitely generated groups of MTs that are discrete in the group of all MTs. Any MT is a map $z \rightarrow \frac{az+b}{cz+b}$ with parameters $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. We want to prove that no system of purely rational MTs exist such that it generates a redundant number system.
It is equivanent to showing that some system of diophantic equations in *eight* variables has no solution. We would like to ask the auditorium for some ideas how to show it.

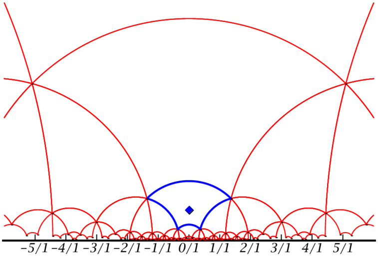
Hyperbolic plane

Poincaré models:

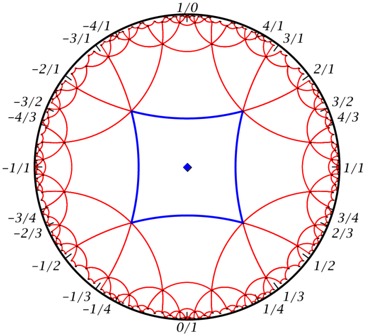
- $\mathbb{U} = \{z \in \mathbb{C} | \Im z > 0\}$ (upper half-plane) — for computations
- metric: $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$, where $z = x + iy$
- $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ (unit complex disc) — for visualization
- isometry $d : \mathbb{U} \to \mathbb{D}$, $d(z) = \frac{iz+1}{z+i}$
- isometry is conformal (preserves angles)
- boundary: $\partial\mathbb{U} = \mathbb{R} \cup \{\infty\}$, $\partial\mathbb{D} = \{z \in \mathbb{C} | |z| = 1\}$

Hyperbolic plane

upper half-plane $\mathbb{U}$



unit disc $\mathbb{D}$



Möbius transformations

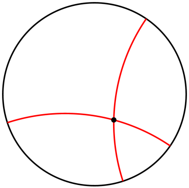
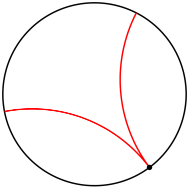
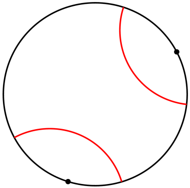
Orientation-preserving Möbius transformation (MT) is
$M_{\mathbf{A}} : \mathbb{U} \to \mathbb{U}$,
$M_{\mathbf{A}}(z) = \frac{az+b}{cz+d}$, where $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ and $\det \mathbf{A} > 0$.

Property

- $M_{\mathbf{AB}} = M_{\mathbf{A}} \circ M_{\mathbf{B}}$,
- $M_{\lambda\mathbf{A}} = M_{\mathbf{A}}$ *for* $\lambda \in \mathbb{R} \setminus \{0\}$,
- *M is conformal isometry (with respect to hyperbolic metric).*

Trace of MT: $\operatorname{tr}^2 M_{\mathbf{A}} = \frac{\operatorname{tr}^2 \mathbf{A}}{\det \mathbf{A}} = \frac{(a+d)^2}{ad-bc}$

Möbius transformations types

Type:	elliptic	parabolic	hyperbolic
Fixed points:	one in $\mathbb{U}$	one in $\partial\mathbb{U}$	two in $\partial\mathbb{U}$
Trace: $\mathrm{tr}^2\, M$	$\in [0, 4)$	$= 4$	$\in (4, \infty)$
Example:			

Angle of rotation **rot** M of elliptic MT: satisfies
$\mathrm{tr}^2\, M = 4\cos^2 \frac{\mathbf{rot}\, M}{2}$

Fuchsian groups and Möbius number systems

A group G of MTs is Fuchsian, if it is discrete, i.e. its elements do not accumulate at identity.

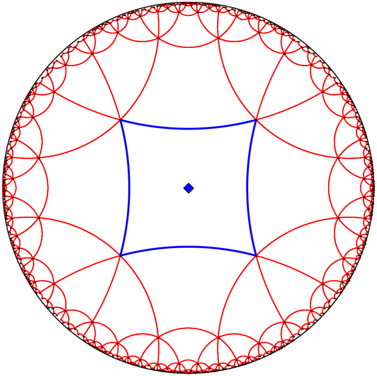
Proposition

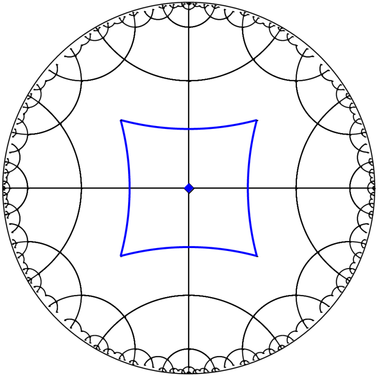
Elliptic MT M has finite order (hence group $\{M^k\}_{k\in\mathbb{Z}}$ is Fuchsian) iff $\mathbf{rot}\, M \in \pi\mathbb{Q}$.

A fundamental domain of G is an area $\mathbb{F} \subset \mathbb{U}$ such that its G-images tesselate $\mathbb{U}$.

Example — $(4, 6)$-square system

Group generators $\quad M_0(z) = \left(2 + 1/\sqrt{3}\right)z, \quad M_1(z) = \frac{z\sqrt{3}+1}{z+\sqrt{3}}$

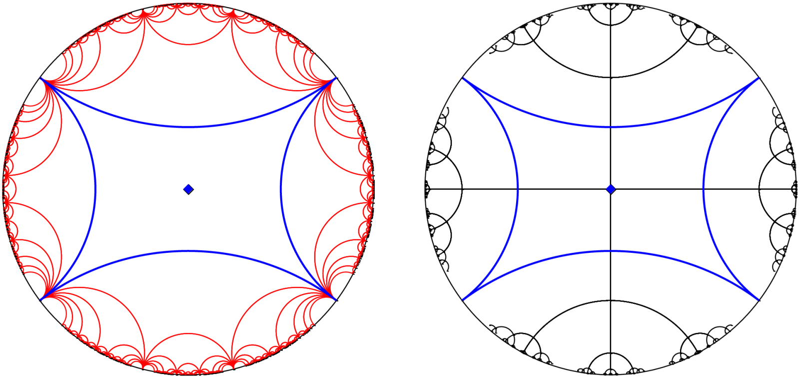




- $+$ redundant
- $+$ many group identities
- $-$ irrational

Example — $(4,\infty)$-rectangular system

Group generators $\quad M_0(z) = z/4, \quad M_1(z) = \frac{5z+4}{4z+5}$



- – not redundant (unbounded fundamental domain)
- – only trivial group identities
- + rational

Our interest: Rational & with bounded fund. dom.

Question: Exists a Fuchsian group of rational transformations with bounded fundamental domain?

Likely answer: No, it does not.

Proposition

Rational elliptic MT M has finite order iff $\operatorname{tr}^2 M \in \mathbb{N}_0$.

Proof:

- The only angles $\theta \in \pi\mathbb{Q}$ such that $\cos\theta \in \mathbb{Q}$ are $\theta \in \{0, \pm\pi/3, \pm\pi/2, \pm 2\pi/3, \pi\} + 2\pi\mathbb{Z}$.
- We get $\operatorname{tr}^2 M = 4\cos^2\theta/2 = 2(1+\cos\theta) \in \{0,1,2,3\}$. $\square$

Our interest: Rational & with bounded fund. dom.

Hypothesis

Let M_1, M_2 be rational elliptic transformations with $\operatorname{tr}^2 M_i \in \{0, 1, 2, 3\}$ that have no common fixed point, and $M_1 \circ M_2$ is elliptic as well. Then $\operatorname{tr}^2(M_1 \circ M_2) \notin \{0, 1, 2, 3\}$.

Main idea: Some diophantic equation have no solution.

Hypothesis

Let G be a Fuchsian group with bounded fundamental domain. Then there exist $M_1, M_2 \in G$ elliptic with no common fixed point such that $M_1 \circ M_2$ is elliptic.

Main idea: "Corners" of a specific fundamental domain (called Ford f.d.) are fixed points of elliptic transformations.

Equations

$$M_1 \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M_2 \sim \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M_1 M_2 \sim \left(\begin{smallmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{smallmatrix}\right)$$

$$\mathrm{tr}^2\, M_1 M_2 = \frac{(aA + bC + cB + dD)^2}{(ad - bc)(AD - BC)} \in \{0, 1, 2, 3\} \tag{1}$$

$$\mathrm{tr}^2\, M_1 = \frac{(a + d)^2}{ad - bc} \in \{0, 1, 2, 3\} \tag{2}$$

$$\mathrm{tr}^2\, M_2 = \frac{(A + D)^2}{AD - BC} \in \{0, 1, 2, 3\} \tag{3}$$

Different fixed points: $$\begin{aligned} \phi_1 &= c(A - D) - (a - d)C \neq 0 \\ \phi_2 &= b(A - D) - (a - d)B \neq 0 \end{aligned} \tag{4}$$

Found "nearly-solutions" (by PC)

$(a,b,c,d) \in \mathbb{Z}^4$	$(A,B,C,D) \in \mathbb{Z}^4$	$\mathrm{tr}^2\, M_1 = 0..3$	$\mathrm{tr}^2\, M_2 = 0..3$	$\mathrm{tr}^2\, M_1 M_2 = 0..3$	$\phi_1 \neq 0$	$\phi_2 \neq 0$
$(6,-3,7,-3)$	$(5,-4,8,-5)$	0	3	$7/3$	2	6
$(1,10,-5,-1)$	$(0,-1,1,1)$	0	1	4	3	-8
$(4,-1,4,2)$	$(0,1,-4,2)$	3	1	3	0	0
$(11,-7,13,19)$	$(-4,-7,13,4)$	0	3	1	0	0
$(-1,5,-1,3)$	$(2,-5,1,-2)$	2	2	0	0	0

Equations

- $t, T, \tau \in \{0, 1, 2, 3\}$ parameters
- cases $\quad t, T \neq 0 \quad$ or $\quad t, T = 0$
- $\frac{(a+d)^2}{ad-bc} = t \quad \Longrightarrow \quad \frac{1}{ad-bc} = \frac{t}{(a+d)^2} \quad$ or $\quad d = -a$
- $\frac{(A+D)^2}{AD-BC} = T \quad \Longrightarrow \quad \frac{1}{AD-BC} = \frac{T}{(A+D)^2} \quad$ or $\quad D = -A$
- $\frac{(aA+bC+cB+dD)^2}{(ad-bc)(AD-BC)} = \tau$
- $\Longrightarrow \quad tT(aA+bC+cB+dD)^2 = \tau(a+d)^2(A+D)^2$
- $\phi_1 = c(A-D)-(a-d)C \neq 0$
$\phi_2 = b(A-D)-(a-d)B \neq 0$