Möbius number systems with discrete groups

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Abstract

We study number systems generated by Möbius transformations (MT) of the hyperbolic plane $\mathbb{U} = \{z \in \mathbb{C} | \Im z \geq 0\}$. We are concerned about finitely generated groups of MTs that are discrete in the group of all MTs. Any MT is a map $z \to \frac{az+b}{cz+b}$ with parameters $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. We will try to answer the question of existence of a discrete group of MTs such that all its elements have rational parameters and corresponding number system is covergent.

Poincaré models:

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- metric: $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$, where $z = x + iy$

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- $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ (unit complex disc) — for visualization
- isometry $d : \mathbb{U} \to \mathbb{D}$, $d(z) = \frac{iz+1}{z+i}$
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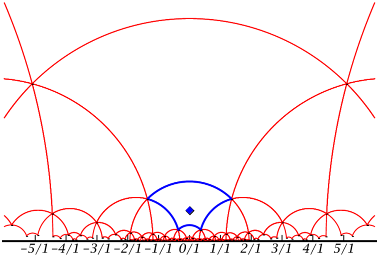
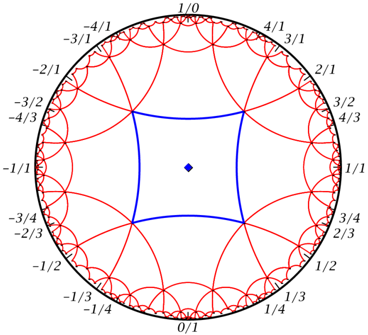
- boundary: $\partial\mathbb{U} = \mathbb{R} \cup \{\infty\}$, $\partial\mathbb{D} = \{z \in \mathbb{C} | |z| = 1\}$

Hyperbolic plane

upper half-plane $\mathbb{U}$



unit disc $\mathbb{D}$



Möbius transformations

Orientation-preserving Möbius transformation (MT) is $M_{\mathbf{A}} : \mathbb{U} \to \mathbb{U}$, $M_{\mathbf{A}}(z) = \frac{az+b}{cz+d}$, where $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ and $\det \mathbf{A} > 0$.

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Property

- $M_{\mathbf{AB}} = M_{\mathbf{A}} \circ M_{\mathbf{B}}$,
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Trace of MT: $\operatorname{tr}^2 M_{\mathbf{A}} = \frac{\operatorname{tr}^2 \mathbf{A}}{\det \mathbf{A}} = \frac{(a+d)^2}{ad-bc}$

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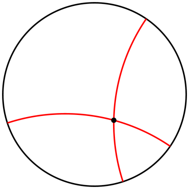
Example:			

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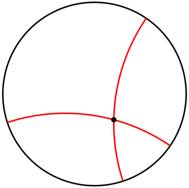
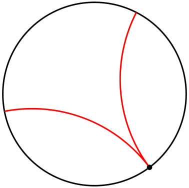
Example:			

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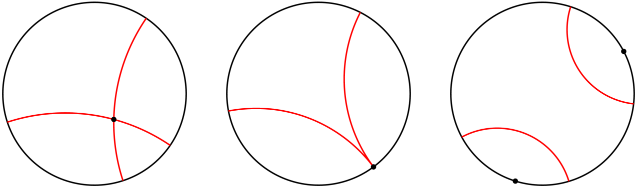
Example:			

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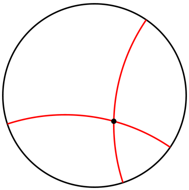
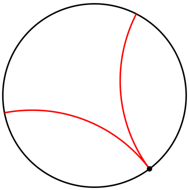
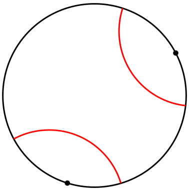
Example:			

Angle of rotation **rot** M of elliptic MT: satisfies $\operatorname{tr}^2 M = 4\cos^2 \frac{\mathbf{rot}\, M}{2}$

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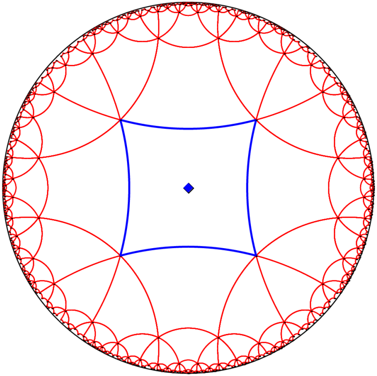
A fundamental domain of G is an area $\mathbb{F} \subset \mathbb{U}$ such that its G-images tesselate $\mathbb{U}$.

Example — (4, 6)-square system

Group generators $M_0(z) = \left(2 + 1/\sqrt{3}\right)z, \quad M_1(z) = \frac{z\sqrt{3}+1}{z+\sqrt{3}}$



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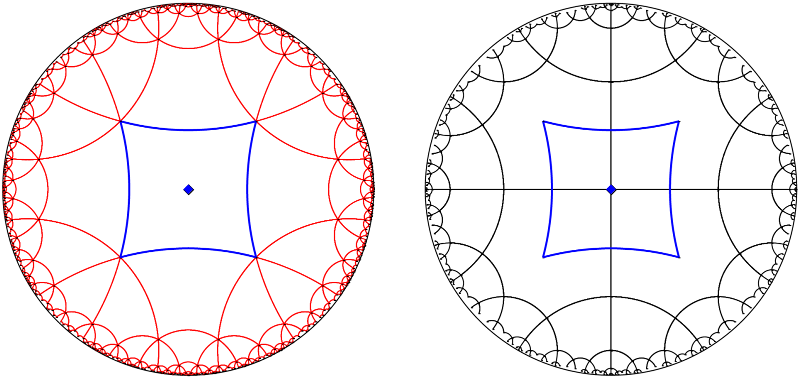

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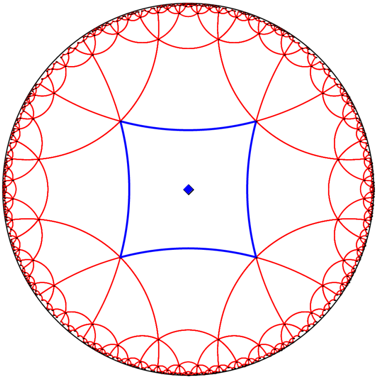
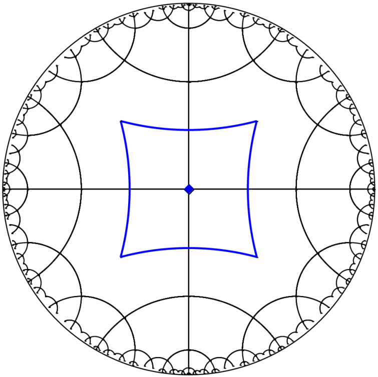
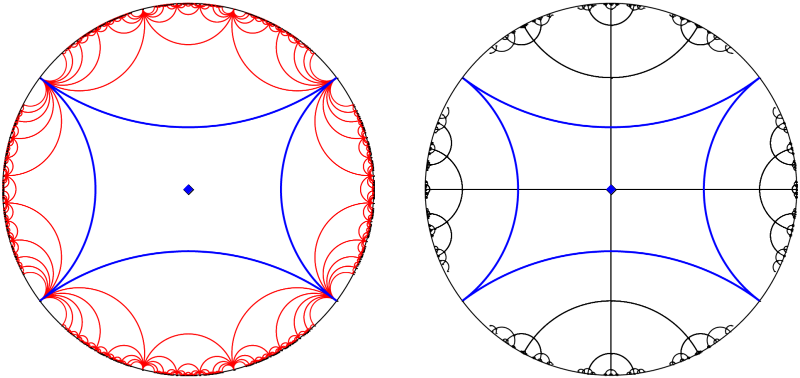




+ redundant
+ many group identities
– irrational

Example — $(4, \infty)$-rectangular system

Group generators $\quad M_0(z) = z/4, \quad M_1(z) = \frac{5z+4}{4z+5}$



— not redundant (unbounded fundamental domain)

— only trivial group identities

+ rational

Our interest: Rational & with bounded fund. dom.

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- We get $\operatorname{tr}^2 M = 4\cos^2\theta/2 = 2(1+\cos\theta) \in \{0, 1, 2, 3\}$. $\square$

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Hypothesis

Let M_1, M_2 be rational elliptic transformations with $\mathrm{tr}^2 M_i \in \mathbb{N}_0$ that have no common fixed point. Then $\mathrm{tr}^2(M_1 \circ M_2) \notin \mathbb{N}_0$.

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Main idea: "Corners" of a specific fundamental domain (called Ford f.d.) are fixed points of elliptic transformations.

Redundant × Rational