Morphisms preserving the set of words coding three interval exchange

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Combinatorics on words

- alphabet $\{0,1\}$, $\{A,B,C\}$
- (right) infinite word $\boldsymbol{u} = u_0 u_1 u_2 \cdots$
- finite word $w = w_0 w_1 \cdots w_{n-1}$, length n
- morphism φ on words, $\varphi(uv) = \varphi(u)\varphi(v)$, given by the images of the letters
- incidence matrix of a morphism:

$$(\mathbf{M}_\varphi)_{ab} = |\varphi(a)|_b$$

Example

Morphism η over $\{A,B,C\}$, $\eta(A) = A$, $\eta(B) = CAC$, $\eta(C) = C$

Incidence matrix $\mathbf{M}_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

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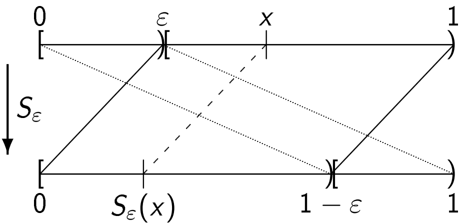
Sturmian words

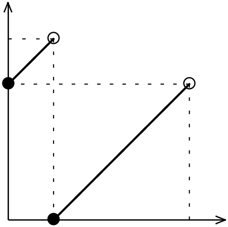
Many ways how to define Sturmian words — infinite $\boldsymbol{u}$ over $\{0,1\}$ is Sturmian, iff,

1. $\boldsymbol{u}$ is aperiodic with minimal complexity
2. $\boldsymbol{u}$ codes irrational exchange of 2 intervals
3. $\boldsymbol{u}$ is aperiodic and balanced
4. $\boldsymbol{u}$ is irrational mechanical
5. $\boldsymbol{u}$ is an irrational cutting sequence
6. ...

2-interval exchange

2-interval exchange transformation S_ε, $\quad \varepsilon \in (0,1)$





2iet word $\mathbf{s}_{\varepsilon,\rho} = s_0 s_1 s_2 \cdots$, $\quad \rho \in [0,1)$, $\quad$ Sturmian for $\varepsilon \notin \mathbb{Q}$

$$s_i = \begin{cases} 0 & \text{if } S_\epsilon^i(\rho) \in [0,\varepsilon), \\ 1 & \text{if } S_\epsilon^i(\rho) \in [\varepsilon,1). \end{cases}$$

Sturmian morphism φ, $\quad \varphi(u)$ is a Sturmian word for every u Sturmian

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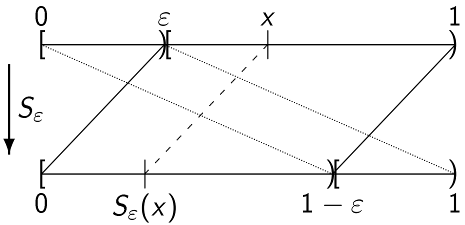

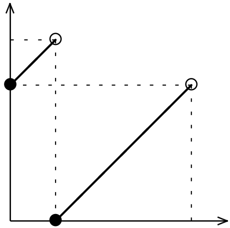

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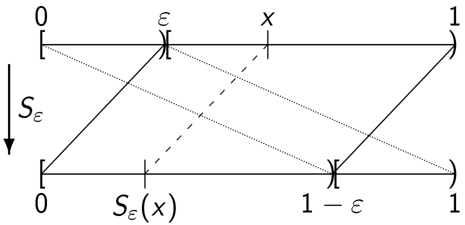

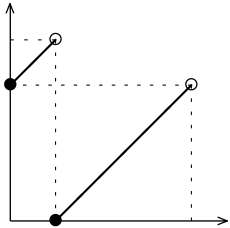

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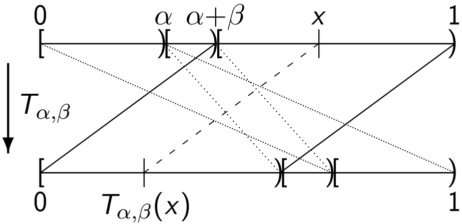
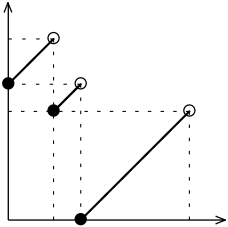
3-interval exchange

3-interval exchange transformation $T_{\alpha,\beta}$, $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$





3iet word $\mathbf{s}_{\alpha,\beta,\rho} = s_0 s_1 s_2 \cdots$, $\rho \in [0,1)$

$$s_i = \begin{cases} A & \text{if } T^i_{\alpha,\beta}(\rho) \in [0,\alpha) \\ B & \text{if } T^i_{\alpha,\beta}(\rho) \in [\alpha, \alpha+\beta) \\ C & \text{if } T^i_{\alpha,\beta}(\rho) \in [\alpha+\beta, 1). \end{cases}$$

3iet-preserving morphism η, $\eta(v)$ is a 3iet word for every v 3iet word

Amicability

- Amicable words: $u \propto v$ ... Example:

$$u = 0\ \ 01\ \ 0\ \ 1\ \ 0\ \ 01\ \ \cdots$$
$$v = 0\ \ 10\ \ 0\ \ 1\ \ 0\ \ 10\ \ \cdots$$

- Ternarization $\quad \mathrm{ter}(u, v) = A\ \ B\ \ A\ \ C\ \ A\ \ B\ \ \cdots$
- Amicable morphisms: $\varphi \propto \psi$, iff, $(\forall u, v)(u \propto v \implies \varphi(u) \propto \psi(v))$
- Ternarization of amicable morphisms $\{A, B, C\} \to \{A, B, C\}$

Proposition (Arnoux, Berthé, Masáková, Pelantová)

Let u, v be infinite words over $\{0, 1\}$, $u \propto v$. Then

$$\mathrm{ter}(u, v) \text{ is a 3iet word} \iff u, v \text{ are Sturmian words.}$$

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Results

Theorem

Every ternarization of amicable Sturmian morphisms is (globally) 3iet-preserving.

Theorem

A matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ *is the incidence matrix of the ternarization of a pair of amicable Sturmian morphisms* $\iff$ *there exist matrix* $\mathbf{A} = \begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$ *and numbers* b_0, b_1 *such that*

(a) $\left| \frac{b_0(p_1+q_1) - b_1(p_0+q_0)}{p_0+q_0+p_1+q_1} \right| < 1,$

(b) $\frac{1-\Delta}{2} \leq b_0 + b_1 \leq \min\{p_0+p_1, q_0+q_1\} - \frac{\Delta+1}{2},$

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Idea of the proofs: rational interval exchange

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Conclusions

1. We desribed a subset of the set 3iet-preserving morphisms – the ternarizations
2. We know which matrices are matrices of ternarizations
3. We assume that the set of all 3iet-preserving morphisms is "not much larger"

Conjecture

There exist four 3iet-preserving morphisms $\psi_1, \dots, \psi_4$ such that for all 3iet-preserving morphisms η, one of $\eta \circ \psi_1, \dots, \eta \circ \psi_4$ is a ternarization of amicable Sturmian morphisms.

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