Morphisms preserving the set of words coding three interval exchange

Tomáš Hejda

Department of Mathematics FNSPE,
Czech Technical University in Prague

tohe@centrum.cz

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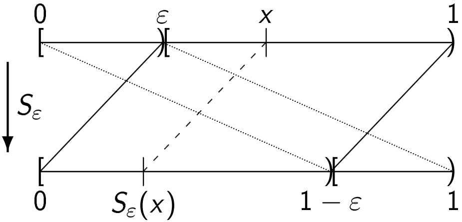
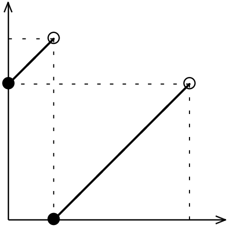
Combinatorics on words

- Alphabet $\{0, 1\}$, $\{A, B, C\}$
- (Right) infinite word $\boldsymbol{u} = u_0 u_1 u_2 \cdots$
- Finite word $w = w_0 w_1 \cdots w_{n-1}$, length n
- Morphism φ on words, $\varphi(uv) = \varphi(u)\varphi(v)$,
  given by the images of the letters

2-interval exchange

- 2-interval exchange transformation S_ε, $\quad \varepsilon \in (0,1)$





- Sturmian word $s_{\varepsilon,x_0} = s_0 s_1 s_2 \cdots$, $\quad x_0 \in [0,1)$, $\quad \varepsilon \notin \mathbb{Q}$

$$s_i = \begin{cases} 0 & \text{if } S_\varepsilon^i(x_0) \in [0,\varepsilon), \\ 1 & \text{if } S_\varepsilon^i(x_0) \in [\varepsilon,1). \end{cases}$$

or partition $(0,1] = (0,\varepsilon] \cup (\varepsilon,1]$

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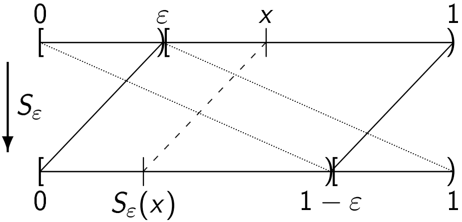

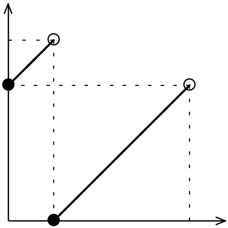

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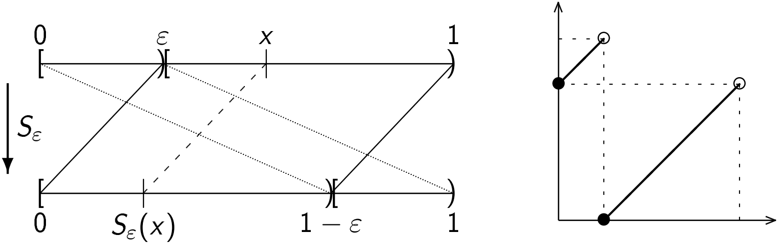

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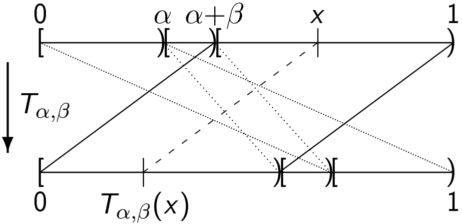
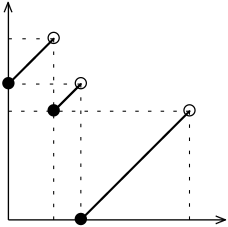
3-interval exchange

- 3-interval exchange transformation $T_{\alpha,\beta}$, $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$





- 3iet word $s_{\alpha,\beta,x_0} = s_0 s_1 s_2 \cdots$, $x_0 \in [0,1)$, $\frac{1-\alpha}{1+\beta} \notin \mathbb{Q}$

$$s_i = \begin{cases} A & \text{if } T^i_{\alpha,\beta}(x_0) \in [0,\alpha) \\ B & \text{if } T^i_{\alpha,\beta}(x_0) \in [\alpha,\alpha+\beta) \\ C & \text{if } T^i_{\alpha,\beta}(x_0) \in [\alpha+\beta,1). \end{cases}$$

or partition $(0,1] = (0,\alpha] \cup (\alpha,\alpha+\beta] \cup (\alpha+\beta,1]$

Sturmian and 3iet-preserving morphisms

- Binary morphism φ is Sturmian
  if $\varphi(\boldsymbol{u})$ is a Sturmian word for every $\boldsymbol{u}$ Sturmian
- **Example:** Fibonacci morphism $\varphi(0) = 01, \quad \varphi(1) = 0$

- Ternary morphism η is 3iet-preserving
  if $\eta(\boldsymbol{u})$ is a 3iet word for every 3iet word $\boldsymbol{u}$
- **Example:** $\eta(A) = B, \quad \eta(B) = CAC, \quad \eta(C) = C$

- Incidence matrix of a morphism:

$$(\mathbf{M}_\varphi)_{ab} = |\varphi(a)|_b$$

- **Example:** $\mathbf{M}_\varphi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{M}_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

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Sturmian	**3iet-preserving**
Monoid of morphisms	
3 generators	not finitely generated
Incidence matrices	
N.C.: $\det \mathbf{M} = \pm 1$	N.C.: $\det \mathbf{M} = \pm 1$ or $\det \mathbf{M} = 0$
S.C.: $\det \mathbf{M} = \pm 1$	S.C.: see Theorem III below
Morphisms to matrix	
exactly $\|\mathbf{M}\| - 1$ morphisms	from $\mathcal{O}(1)$ up to $\mathcal{O}(\|\mathbf{M}\|)$ morphisms
conjugates of one morphism	conjugates of more?

- Amicable Sturmian words/factors: $u \propto v$ ... Example:

$$u = 0\ \ 01\ \ 0\ \ 1\ \ 0\ \ 01\ \ \cdots$$
$$v = 0\ \ 10\ \ 0\ \ 1\ \ 0\ \ 10\ \ \cdots$$

- Ternarization $\quad \mathrm{ter}(u, v) = A\ \ B\ \ A\ C\ A\ \ B\ \ \cdots$
- **Observation:** Parikh vectors or u and v coincide (finite case)
  The words $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same slope (infinite case)

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- Amicable Sturmian morphisms: $\varphi \propto \psi$ if

$$\varphi(0) \propto \psi(0), \qquad \varphi(1) \propto \psi(1), \quad \text{and} \quad \varphi(01) \propto \psi(10)$$

- **Observation:** Incidence matrices of φ and ψ coincide.

- **Example:**

$$\begin{array}{lll} \varphi(0) = 010, & \varphi(1) = \underline{01}, & \varphi(01) = \underline{01}\,\underline{001} \\ \psi(0) = 010, & \psi(1) = 10, & \psi(10) = 10010 \\ \eta(A) = ACA, & \eta(C) = B, & \eta(B) = BAB \end{array}$$

- Ternarization of amicable morphisms $\eta = \mathrm{ter}(\varphi, \psi)$

$$\eta(A) = \mathrm{ter}(\varphi(0), \psi(0))$$
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Proposition (Arnoux, Berthé, Masáková, Pelantová)

Let $\mathbf{u}, \mathbf{v}$ be Sturmian words, $\mathbf{u} \propto \mathbf{v}$. Then $\operatorname{ter}(\mathbf{u}, \mathbf{v})$ is a 3iet word. Moreover, every 3iet word is a ternarization of a pair of amicable Sturmian words.

Proposition

Let $\mathbf{u} \propto \mathbf{v}$ be amicable words and $\varphi \propto \psi$ amicable morphisms. Then $\varphi(\mathbf{u}) \propto \psi(\mathbf{v})$ and

$$\operatorname{ter}(\varphi, \psi)(\operatorname{ter}(\mathbf{u}, \mathbf{v})) = \operatorname{ter}(\varphi(\mathbf{u}), \psi(\mathbf{v})).$$

Properties of ternarizations I.

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Properties of ternarizations II.

Theorem (I.)

1. *Every ternarization* $\eta = \mathrm{ter}(\varphi, \psi)$ *of a pair of amicable Sturmian morphisms* $\varphi \propto \psi$ *is 3iet-preserving*
2. *Ternarizations satisfy* $\mathrm{ter}(\varphi, \psi) \circ \mathrm{ter}(\varphi', \psi') = \mathrm{ter}(\varphi \circ \varphi', \psi \circ \psi')$
3. *All incidence matrices of them have* $\det \mathbf{M} = +1$

- There exist 3iet-preserving morphisms that are not ternarizations
- Example 1: $\eta(A) = C, \quad \eta(B) = B, \quad \eta(C) = A$
- Example 2: $\eta(A) = B$, $\eta(B) = CAC$, $\eta(C) = C$

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Number of amicable pairs

Theorem (II.)

Let $\mathbf{A} = \begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$*. Then the number of amicable pairs of Sturmian morphisms* $\varphi \propto \psi$ *with* $\mathbf{M}_\varphi = \mathbf{M}_\psi = \mathbf{A}$ *is equal to*

$$m(N-1) + \frac{m}{2}(\det \mathbf{A} - m),$$

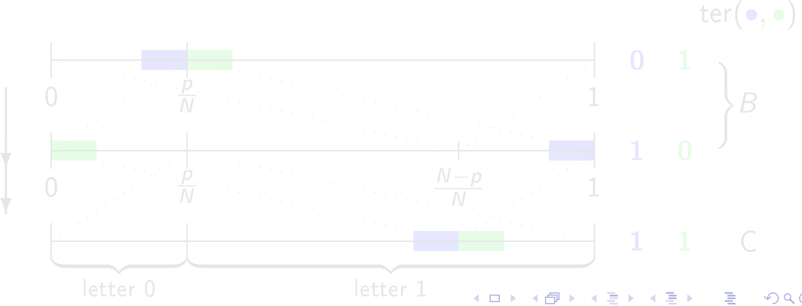
where $m = \min\{p_0 + p_1, q_0 + q_1\}$ *and* $N = p_0 + q_0 + p_1 + q_1$*.*

Theorem II.

Ideas of the proof (for the case $\det \mathbf{A} = +1$)

- Notions: $p = p_0 + p_1$, $N = p_0 + p_1 + q_0 + q_1$, $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$
- Let φ be a Sturmian morphism with incidence matrix $\mathbf{A}$. Then $\varphi(01)$ codes 2iet with rational slope p/N and start point k/N for some $k \in \{0, \dots, N-1\}$
- Because $p \perp N$, the orbit satisfies

$$\{S^i(k/N)\}_{i=0}^{N-1} = \{0, 1/N, \dots, (N-1)/N\}$$



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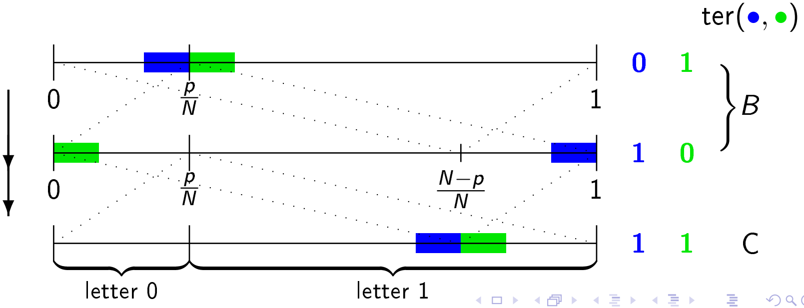

Theorem III.

Theorem

A matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ *is the incidence matrix of the ternarization of a pair of amicable Sturmian morphisms* $\iff$ *there exist matrix* $\mathbf{A} = \begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix} \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$ *and numbers* b_0, b_1 *such that*

(a) $\left| \frac{b_0(p_1+q_1)-b_1(p_0+q_0)}{p_0+q_0+p_1+q_1} \right| < 1,$

(b) $\frac{1-\Delta}{2} \leq b_0 + b_1 \leq \min\{p_0 + p_1, q_0 + q_1\} - \frac{\Delta+1}{2},$

(c) $\mathbf{B} = \mathbf{P} \begin{pmatrix} \mathbf{A} & \begin{matrix} b_0 \\ b_1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \Delta \end{pmatrix} \mathbf{P}^{-1}$, *where* $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Ideas of the proof:

1. Rational 2-interval exchange
2. Numbers p and N are co-prime
3. Congruence modulo N

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Open problem — Conjectures

Conjecture

There exist four 3iet-preserving morphisms $\xi_1, \dots, \xi_4$ such that for all 3iet-preserving morphisms η we have:

1. *either* $\det \mathbf{M}_\eta = 0$,
2. *or* $\exists i \in \{1, \dots 4\}$ *such that* $\xi_i \circ \eta$ *is a ternarization of amicable Sturmian morphisms.*

Conjecture

Let η be a 3iet-preserving morphism, and η' a morphism over $\{A, B, C\}^*$ satisfying $\mathbf{M}_\eta = \mathbf{M}_{\eta'}$. Then η' is 3iet-preserving iff η' is conjugate to η, i.e., there exists a word $x \in \{A, B, C\}^*$ such that

$$(\forall y \in \{A, B, C\}^*)(\eta(y)x = x\eta'(y))$$

$$\text{or} \quad (\forall y \in \{A, B, C\}^*)(x\eta(y) = \eta'(y)x).$$

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What would the conjectures mean?

1. Full characterization of matrices of 3iet-preserving morphisms.
2. Idea how to generate all 3iet-preserving morphisms to a matrix (using standard morphisms).

What would they not mean?

3. Generators of monoid of 3iet-preserving morphisms.
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References

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