Arithmetic Complexity of Sturmian Words

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based on work of J. Cassaigne and A. Frid

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Combinatorics on Words, Hojsova Straz 2010

Notation

- alphabet $\{0,1\}$
- (right) infinite word $\mathbf{s} = s_0 s_1 s_2 \cdots$
- finite word $w = w_0 w_1 \cdots w_{n-1} w_n$, length $n+1$
- fractional part of $x \in \mathbb{R}$ is $\{\!\{x\}\!\} = x - \lfloor x \rfloor$.

Complexity Functions

- factor complexity $\mathcal{C}_{\boldsymbol{u}}(n+1) = \#$ of "subword" factors

$$\mathcal{L}_{\boldsymbol{u}}(n+1) = \{u_k u_{k+1} u_{k+2} \cdots u_{k+n} | k \geq 0\}$$

- Abelian complexity $\mathcal{C}^{ab}_{\boldsymbol{u}}(n+1) = \#$ of Parikh vectors

$$\{(|w|_0, |w|_1) \big| w \in \mathcal{L}_{\boldsymbol{u}}(n+1)\}$$

- arithmetic complexity $\mathcal{C}^{ar}_{\boldsymbol{u}}(n+1) = \#$ of arithmetic factors

$$\mathcal{A}_{\boldsymbol{u}}(n+1) = \{u_k u_{k+d} u_{k+2d} \cdots u_{k+nd} | k \geq 0, d \geq 1\}$$

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Sturmian Words

Many ways how to define Sturmian words — infinite $\boldsymbol{u}$ is Sturmian, iff,

1. factor complexity satisfies $\quad \mathcal{C}_{\boldsymbol{u}}(n) = n + 1 \quad \forall n \geq 0$
2. $\boldsymbol{u}$ is aperiodic and balanced
3. $\boldsymbol{u}$ is aperiodic and Abelian complexity satisfies
$$\mathcal{C}^{ab}_{\boldsymbol{u}}(n) = 2 \quad \forall n \geq 1$$
4. $\boldsymbol{u}$ is a rotation word with irrational slope α
   - lower rotation word $s_\alpha(\rho) = s_0 s_1 \cdots, \quad \alpha, \rho \in [0,1)$
   $$s_k = \begin{cases} 1 & \text{if } \{(k+1)\alpha + \rho\} < \alpha, \\ 0 & \text{otherwise,} \end{cases}$$
   - upper rotation word $s'_\alpha(\rho)$ — $\leq$ instead of $<$
5. $\boldsymbol{u}$ codes irrational 2iet
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Theorem

Let $\boldsymbol{u}$ be Sturmian word. Then its arithmetic complexity satisfies for all $n \geq 1$

$$\frac{n^3}{4\pi^2} + O(n^2) \leq \mathcal{C}^{ar}_{\boldsymbol{u}}(n) \leq \left(\frac{1}{6} + \frac{1}{\pi^2}\right) n^3 + O(n^2).$$

We prove only upper bound (lower bound as well by Frid).

Bounds for Arithmetic Complexity of Sturmian Words

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Sturmian Words as Mechanical Words

- lower rotation word $\mathbf{s}_\alpha(\rho) = s_0 s_1 \cdots, \quad \alpha, \rho \in [0,1)$

$$s_k = \begin{cases} 1 & \text{if } \{\!\{k\alpha + \rho\}\!\} < \alpha, \\ 0 & \text{otherwise,} \end{cases}.$$

- from now on, we fix $\alpha \in [0,1) \setminus \mathbb{Q}$
- we define $w_\alpha(\beta, \gamma, n) = w_0 \cdots w_n, \quad \beta, \gamma \in [0,1), \quad$ length $n+1$,

$$w_i = \begin{cases} 1 & \text{if } \{\!\{i\beta + \gamma\}\!\} < \alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad 0 \le k \le n$$

Lemma

1. $\mathcal{L}_{s_\alpha(\rho)}(n)$ and $\mathcal{A}_{s_\alpha(\rho)}(n)$ depends only on α (denote $\mathcal{L}_\alpha(n)$, $\mathcal{A}_\alpha(n)$)
2. $w_\alpha(\beta, \gamma, n) \in \mathcal{A}_\alpha \iff \exists k, d : \beta = \{\!\{d\alpha\}\!\}, \gamma = \{\!\{k\alpha + \rho\}\!\}$
3. $\mathcal{A}_\alpha = \bigcup_{\beta,\gamma \in [0,1)} w_\alpha(\beta, \gamma, n)$

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Planar Representation

- $w_\alpha(\beta, \gamma, n) = w_0 \cdots w_n$,

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- planar representation
  - line $y = \beta x + \gamma$
  - closest points below the line
  - sequence of • defines $w_\alpha(\beta, \gamma, n)$
  - **Question (not open):** Can different sequences of • that came from some lines define the same word?

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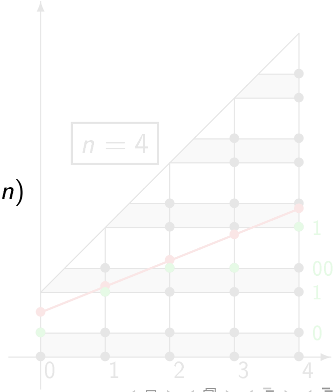

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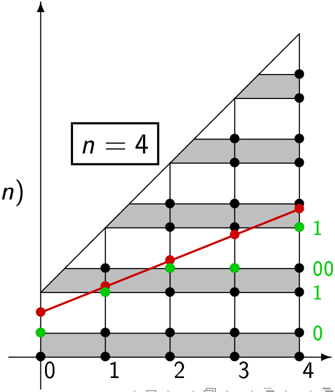

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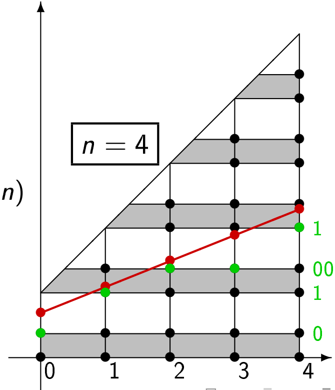

Dual transformation:

- line $l \equiv \; y = \beta x + \gamma$ maps to point $l^* = (\beta, -\gamma)$
- point $p = (\beta, \gamma)$ maps to line $p^* \equiv \; y = \beta x - \gamma$

Lemma

1. *$l^{**} = l$ and $p^{**} = p$.*
2. *Point $p = (a, b)$ is below/above/on line $l \equiv \; y = cx + d \iff$ point $l^* = (c, -d)$ is below/above/on line $p^* \equiv \; y = ax - b$.*
3. *Lines $l_1, \ldots, l_k$ intersect in one point $p \iff$ points $l_1^*, \ldots, l_k^*$ lies on the same line p^*.*

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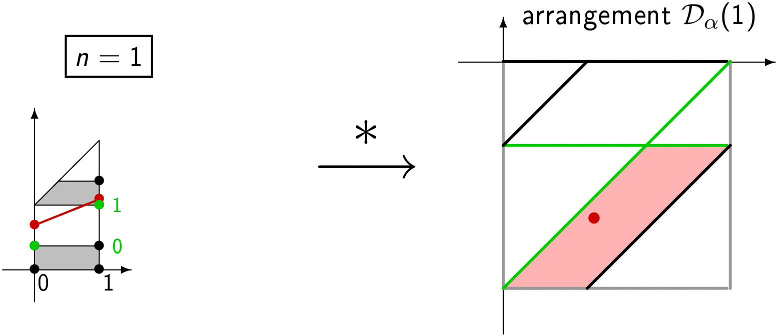

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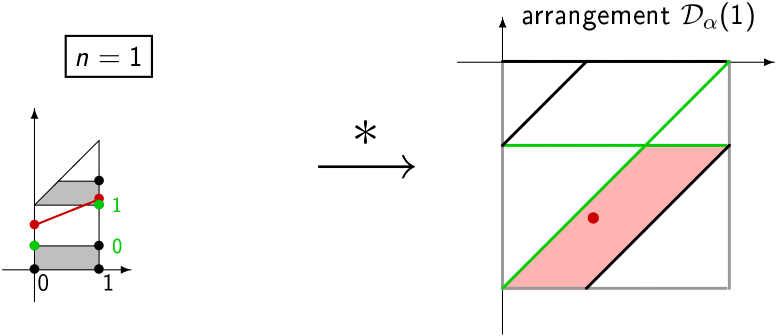

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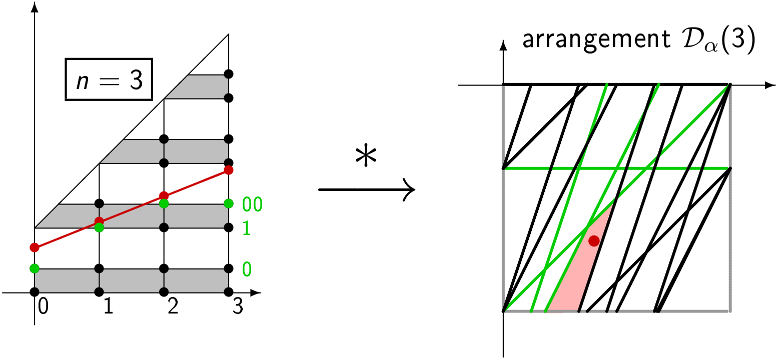

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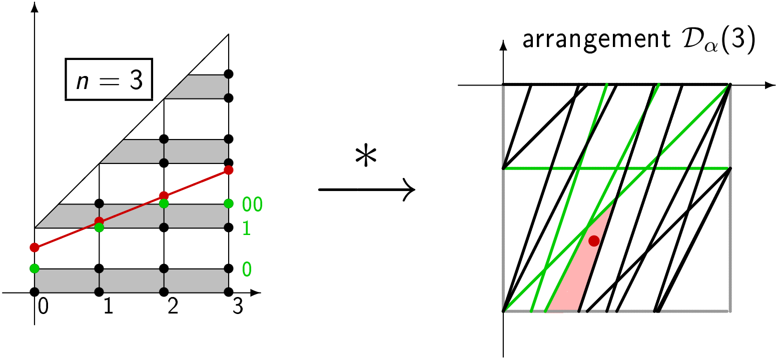

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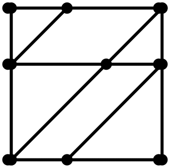
Theorem (Euler's Formula)

Let (V, E) be a planar graph with faces F. Then

$$\#F = \#E - \#V + 1.$$



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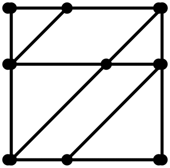

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   - a crossing generates 2 new edges (crossing points are unique)
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what we got	vs.	what Theorem says
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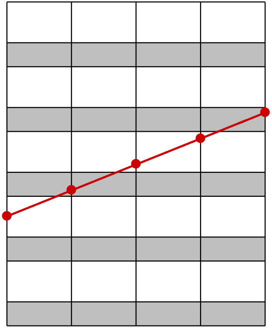

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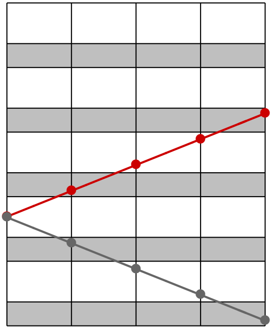

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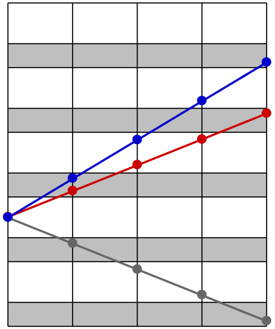

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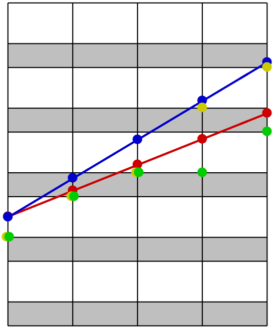

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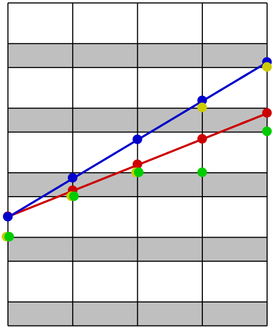

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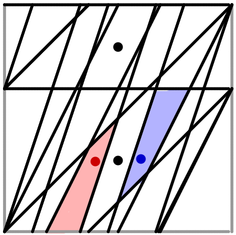

1. The upper bound is independent of α.
2. Both lower and upper bound is $\sim n^3$ (upper is 10.58 larger).
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$$s_\alpha(\beta,\rho), \quad \beta \notin \mathbb{Q}, \quad s_k = \begin{cases} 1 & \text{if } \{\!\{(k+1)\beta+\rho\}\!\} < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

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References

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