#### Arithmetic Complexity of Sturmian Words

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Combinatorics on Words, Hojsova Straz 2010

- alphabet  $\{0,1\}$
- (right) infinite word  $s = s_0 s_1 s_2 \cdots$
- finite word  $w = w_0 w_1 \cdots w_{n-1} w_n$ , length n+1
- fractional part of  $x \in \mathbb{R}$  is  $\{\!\!\{x\}\!\!\} = x \lfloor x \rfloor$ .

factor complexity C<sub>u</sub>(n + 1) = # of "subword" factors
L<sub>u</sub>(n + 1) = {u<sub>k</sub>u<sub>k+1</sub>u<sub>k+2</sub> ··· u<sub>k+n</sub>|k ≥ 0}
Abelian complexity C<sup>ab</sup><sub>u</sub>(n + 1) = # of Parikh vectors

) arithmetic complexity  $\mathcal{C}^{ar}_{m{u}}(n+1)=\#$  of arithmetic factors

$$\mathcal{A}_{\boldsymbol{u}}(n+1) = \{u_k u_{k+d} u_{k+2d} \cdots u_{k+nd} | k \ge 0, d \ge 1\}$$

**Example:** factors and arit. factors of  $\boldsymbol{u} = (01)^{\omega}$ 

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- (1) factor complexity satisfies  $\mathcal{C}_{\boldsymbol{u}}(n)=n+1 \quad \forall n\geq 0$
- ② *u* is aperiodic and balanced
- ③ u is aperiodic and Abelian complexity satisfies C<sup>ab</sup><sub>u</sub>(n) = 2 ∀n ≥ 1
- ④  $oldsymbol{u}$  is a rotation word with irrational slope lpha
  - $\circ$  lower rotation word  $s_lpha(
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(4) **u** is a rotation word with irrational slope  $\alpha$  $\Rightarrow$  lower rotation word  $s_{\alpha}(\rho) = s_0 s_1 \cdots s_n$   $\alpha, \rho \in [0, 1]$ 

 $i_k = \begin{cases} 1 & \text{if } \{\!\!\{(k+1)\alpha + \rho\}\!\!\} < \alpha, \\ 0 & \text{otherwise,} \end{cases}$ 

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## Bounds for Arithmetic Complexity of Sturmian Words

#### Theorem

Let **u** be Sturmian word. Then its arithmetic complexity satisfies for all  $n \ge 1$ 

$$\frac{n^3}{4\pi^2} + O(n^2) \le C_{u}^{ar}(n) \le \left(\frac{1}{6} + \frac{1}{\pi^2}\right)n^3 + O(n^2).$$

We prove only upper bound (lower bound as well by Frid).

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We prove only upper bound (lower bound as well by Frid).

• lower rotation word  $s_{lpha}(
ho)=s_0s_1\cdots$ ,  $lpha,
ho\in[0,1)$ 

$$s_k = \begin{cases} 1 & \text{ if } \{\!\!\{k\alpha + \rho\}\!\!\} < \alpha, \\ 0 & \text{ otherwise,} \end{cases}$$

• from now on, we fix  $\alpha \in [0, 1) \setminus \mathbb{Q}$ • we define  $w_{\alpha}(\beta, \gamma, n) = w_0 \cdots w_n$ ,  $\beta, \gamma \in [0, 1)$ , length n + 1

$$w_i = \begin{cases} 1 & \text{if } \{\!\{i\beta + \gamma\}\!\} < \alpha, \\ 0 & \text{otherwise,} \end{cases} \quad 0 \le k \le n$$

#### Lemma

Is  $\mathcal{L}_{\mathbf{x}_{\alpha}(\rho)}(n)$  and  $\mathcal{A}_{\mathbf{x}_{\alpha}(\rho)}(n)$  depends only on  $\alpha$  (denote  $\mathcal{L}_{\alpha}(n)$ ,  $\mathcal{A}_{\alpha}(n)$ )
Is  $w_{\alpha}(\beta, \gamma, n) \in \mathcal{A}_{\alpha} \iff \exists k, d : \beta = \{\!\!\{d\alpha\}\!\!\}, \gamma = \{\!\!\{k\alpha + \rho\}\!\!\}$ Is  $\mathcal{A}_{\alpha} = [1_{\alpha} = 0, \infty, w_{\alpha}(\beta, \gamma, n)]$ 

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#### Lemma

 $\begin{array}{l} \bigcirc \mathcal{L}_{\mathbf{s}_{\alpha}(\rho)}(n) \text{ and } \mathcal{A}_{\mathbf{s}_{\alpha}(\rho)}(n) \text{ depends only on } \alpha \text{ (denote } \mathcal{L}_{\alpha}(n), \mathcal{A}_{\alpha}(n)) \\ \bigcirc w_{\alpha}(\beta, \gamma, n) \in \mathcal{A}_{\alpha} \iff \exists k, d : \beta = \{\!\!\{d\alpha\}\!\!\}, \gamma = \{\!\!\{k\alpha + \rho\}\!\!\} \\ \bigcirc \mathcal{A}_{\alpha} = \{\!\!\{l_{\alpha} \mid l_{\alpha} \mid l_$ 

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**1**  $\mathcal{L}_{s_{\alpha}(\rho)}(n)$  and  $\mathcal{A}_{s_{\alpha}(\rho)}(n)$  depends only on  $\alpha$  (denote  $\mathcal{L}_{\alpha}(n)$ ,  $\mathcal{A}_{\alpha}(n)$ ) **2**  $w_{\alpha}(\beta, \gamma, n) \in \mathcal{A}_{\alpha} \iff \exists k, d : \beta = \{\!\{d\alpha\}\!\}, \gamma = \{\!\{k\alpha + \rho\}\!\}$  **3**  $\mathcal{A}_{\alpha} = \bigcup_{\beta, \gamma \in [0,1)} w_{\alpha}(\beta, \gamma, n)$ 

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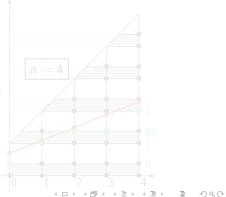
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$$w_{\alpha}(\beta, \gamma, n) = w_0 \cdots w_n$$
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0  $\leq k \leq n$   
• planar representation  
• line  $y = \beta x + \gamma$   
• closest points below the line  
• sequence of • defines  $w_{\alpha}(\beta, \gamma, n)$ 

 Question (not open): Can different sequences of • that came from some lines define the same word?



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Tom Hejda (CTU)

Arithmetic Complexity

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#### Dual transformation:

- line  $l \equiv y = \beta x + \gamma$  maps to point  $l^* = (\beta, -\gamma)$
- point  $p=(eta,\gamma)$  maps to line  $p^*\equiv y=eta x-\gamma$

#### Lemma

1) 
$$l^{**} = l$$
 and  $p^{**} = p$ .

- Point p = (a, b) is below/above/on line l ≡ y = cx + d ⇔ point l\* = (c, -d) is below/above/on line p\* ≡ y = ax - b.
- 3 Lines l<sub>1</sub>,..., l<sub>k</sub> intersect in one point p ↔ points l<sub>1</sub><sup>\*</sup>,..., l<sub>k</sub><sup>\*</sup> lies on the same line p<sup>\*</sup>.

#### Dual transformation:

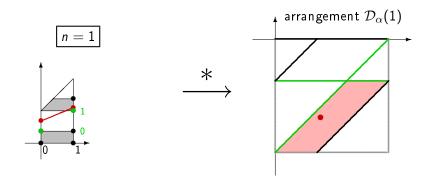
- line  $l\equiv y=eta x+\gamma$  maps to point  $l^*=(eta,-\gamma)$
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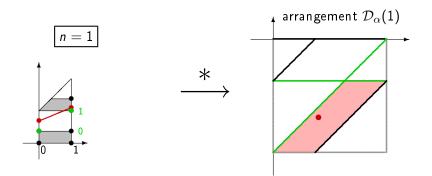
- 2 Point p = (a, b) is below/above/on line  $l \equiv y = cx + d \iff$  point  $l^* = (c, -d)$  is below/above/on line  $p^* \equiv y = ax b$ .
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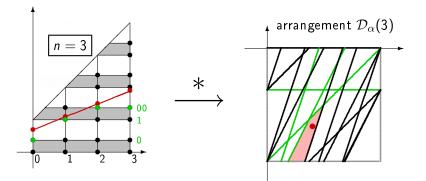


face of arrangement D<sub>α</sub>(n) defines arithmetic factor in A<sub>α</sub>(n + 1)
 it follows: C<sup>ar</sup><sub>α</sub>(n + 1) ≤ # faces of D<sub>α</sub>(n)

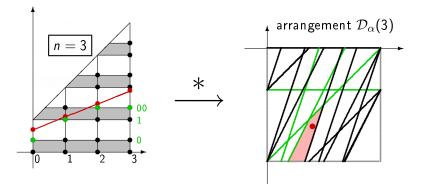
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# Counting Faces of $\mathcal{D}_{\alpha}(n)$

Theorem (Euler's Formula) Let (V, E) be a planar graph with faces F. Then #F = #E - #V + 1.



#### 4 types of vertices:

- (1) "boundary" vertices
- 2 crossings of lines of "0"-type
- 3 crossings of lines of "1"-type
- ④ crossings between both types

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# Counting Faces of $\mathcal{D}_{\alpha}(n)$

- ① "boundary" vertices #new edges – # "boundary" vertices =  $O(\# \text{ lines}) = O(n^2)$ 
  - # new edges # crossings =  $\frac{1}{3}n^3 + O(n^2)$

5 together

$$\#F = \#E - \#V + 1 = \left(\frac{1}{3} + \frac{2}{\pi^2}\right)n^3 + O(n^2)$$

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- "boundary" vertices
   #new edges # "boundary" vertices = O(# lines) = O(n<sup>2</sup>)
   crossings of lines of "0"-type
   Bestel, Pocchiola: # new edges # crossings = <sup>1</sup>/<sub>π<sup>2</sup></sub>n<sup>3</sup> + O(n<sup>2</sup>)
   crossings of lines of "1"-type
   Bestel, Pocchiola: # new edges # crossings = <sup>1</sup>/<sub>π<sup>2</sup></sub>n<sup>3</sup> + O(n<sup>2</sup>)
   crossings between both types
  - lines of both types:  $y = \{\!\{ix\}\!\} 1, \quad y = \{\!\{jx \alpha\}\!\} 1, i, j = 0, \dots, n$
  - equation  $\{x\} 1 = \{x \alpha\} 1$  has |j i| solutions in [0, 1)

• 
$$\sum_{i,j=0}^{n} |j-i| = \frac{1}{3}n(n+1)(n+2) = \#$$
 crossings

- a crossing generates 2 new edges (crossing points are unique)
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   crossings of lines of "0"-type Bestel, Pocchiola: # new edges - # crossings = <sup>1</sup>/<sub>π<sup>2</sup></sub> n<sup>3</sup> + O(n<sup>2</sup>)
   crossings of lines of "1"-type Bestel, Pocchiola: # new edges - # crossings = <sup>1</sup>/<sub>π<sup>2</sup></sub> n<sup>3</sup> + O(n<sup>2</sup>)
   crossings between both types

   lines of both types: y = {[x]} - 1, y = {[x - α]} - 1, i, j = 0, ..., n
  - $\sum_{i,j=0}^{n} |j-i| = \frac{1}{3}n(n+1)(n+2) = \#$  crossings
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$$\#F = \#E - \#V + 1 = \left(\frac{1}{3} + \frac{2}{\pi^2}\right)n^3 + O(n^2)$$

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- ① "boundary" vertices #new edges – # "boundary" vertices =  $O(\# \text{ lines}) = O(n^2)$ 2 crossings of lines of "0"-type Bestel, Pocchiola: # new edges – # crossings =  $\frac{1}{r^2}n^3 + O(n^2)$ 3 crossings of lines of "1"-type Bestel, Pocchiola: # new edges – # crossings =  $\frac{1}{r^2}n^3 + O(n^2)$
- Generalized Construction of the second se
  - lines of both types:  $y = \{ x \} 1, \quad y = \{ x \alpha \} 1,$  $i, j = 0, \ldots, n$
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5 together

$$\#F = \#E - \#V + 1 = \left(\frac{1}{3} + \frac{2}{\pi^2}\right)n^3 + O(n^2)$$

what we got
$$\left(\frac{1}{3} + \frac{2}{\pi^2}\right)n^3 + O(n^2)$$

what Theorem says 
$$\left(rac{1}{6}+rac{1}{\pi^2}
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Question (not open): Can different sequences of • that came from some lines define the same word?

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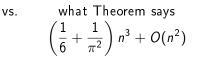
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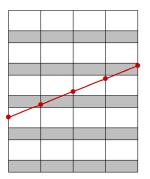
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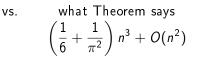
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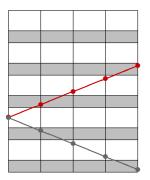
Question (not open): Can different sequences of • that came from some lines define the same word? Answer: Yes, they can.



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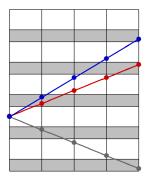
Question (not open): Can different sequences of • that came from some lines define the same word? Answer: Yes, they can.



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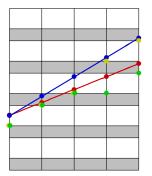
VS.

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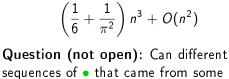
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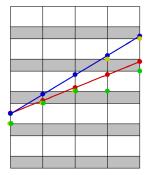
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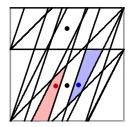
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VS.

#### 1) The upper bound is independent of $\alpha$ .

② Both lower and upper bound is  $\sim n^3$  (upper is 10.58 larger).

3 The upper bound is satisfied for larger set of words than Sturmian:

$$s_{lpha}(eta,
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④ Can be generalized to 3iet with permutation 0 ightarrow 1, 1 ightarrow 2, 2 ightarrow 0.

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- J. Berstel, M. Pocchiola, A geometric proof of the enumeration formula for Sturmian words, Internat. J. Algebra Comput. 3 (1993) 349–355.
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