Counting Amicable Sturmian Morphisms

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Combinatorics on Words, Decin 2009

Outline

Basic Terms

- Alphabet, e.g. $\{0,1\}$
- Finite words, $u \in \{0,1\}^*$, length $|u|$
- Finite + infinite words, $u \in \{0,1\}^\infty$
- Morphisms, e.g. $\varphi : \begin{matrix} 0 \to 01 \\ 1 \to 0 \end{matrix}$
- Morphism matrix, $M_\varphi = \begin{pmatrix} |\varphi(0)|_0 & |\varphi(1)|_0 \\ |\varphi(0)|_1 & |\varphi(1)|_1 \end{pmatrix}$

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Sturmian Words Definitions

Equivalent definitions:

1. Minimal complexity, $C(N) = N + 1$
2. Balanced and aperiodic
3. Two-interval exchange transformation
4. more...

Example (Fibonacci Word)

$$f = 0100101001001010010100100101001001\ldots$$

Factors: ε, $0, 1$, $01, 10, 00$, $010, 100, 001, 101$,
$0100, 1001, 0010, 0101, 1010$, $\ldots$

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Sturmian Morphisms Definition

Morphism is sturmian, if it preserves sturmian words

Example

Identity morphism $I: \begin{matrix} 0 \to 0 \\ 1 \to 1 \end{matrix}$ $\qquad M_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exchange morphism $E: \begin{matrix} 0 \to 1 \\ 1 \to 0 \end{matrix}$ $\qquad M_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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Amicable Words

Example

$$\left.\begin{array}{r} u = 00100101 \\ \mathrm{ter}(u,v) = 0010\,2\,01 \\ v = 00101001 \end{array}\right\rangle \text{1-amicable} \quad u \propto v$$

$$\left.\begin{array}{r} v = 00101001 \\ \mathrm{ter}(v,w) = 0\,2\ \ 2\,0\,2 \\ w = 01010010 \end{array}\right\rangle \text{3-amicable} \quad v \propto w$$

Property

1. $u \propto v \implies u \overset{\text{lex}}{\leq} v$
2. $u \propto v \;\&\; x \propto y \implies ux \propto vy$
3. $u \propto v \;\&\; x \propto y \impliedby ux \propto vy \;\&\; |u| = |v| \;\&\; |u|_0 = |v|_0$

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u, v: 1-amicable $\quad u \propto v$

v, w: 3-amicable $\quad v \propto w$

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Amicable Morphisms

$$\varphi \propto \psi \iff \begin{cases} \varphi(0) \propto \psi(0) \\ \varphi(1) \propto \psi(1) \\ \varphi(01) \propto \psi(10) \end{cases}$$

Example

$$\varphi : \begin{array}{l} 0 \to 001 \\ 1 \to 01001 \end{array} \qquad \psi : \begin{array}{l} 0 \to 010 \\ 1 \to 01010 \end{array}$$

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Objectives and Motivation

1. For given matrix A, $\det A = \pm 1$, count amicable pairs of sturmian morphisms with this matrix.
2. Find, how many of them are b-amicable for $b \in \mathbb{N}$.

Theorem

Let $\varphi, \psi : \{0,1\}^* \mapsto \{0,1\}^*$ be two primitive sturmian morphisms such that $\varphi \propto \psi$. Then morphism $\eta : \{0,1,2\}^* \mapsto \{0,1,2\}^*$,

$$\eta : \begin{aligned} 0 &\to \mathrm{ter}(\varphi(0), \psi(1)) \\ 1 &\to \mathrm{ter}(\varphi(1), \psi(1)) \\ 2 &\to \mathrm{ter}(\varphi(01), \psi(10)) \end{aligned}$$

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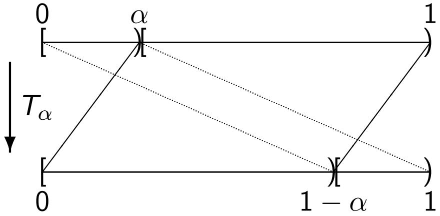
Two-interval exchange transformation — 2iet



Property

- α *rational* — $t_{\alpha,\rho}$ *periodic word*
- α *irrational* — $t_{\alpha,\rho}$ *sturmian word*

Example

For $\alpha = 3/8$ we get

$$(T_\alpha^k(0))_{k=0}^{\infty} = 0, \frac{5}{8}, \frac{2}{8}, \frac{7}{8}, \frac{4}{8}, \frac{1}{8}, \frac{6}{8}, \frac{3}{8}, 0, \frac{5}{8} \dots$$

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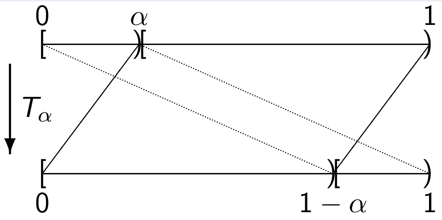

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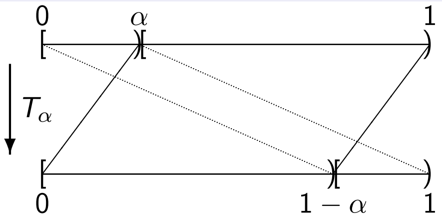

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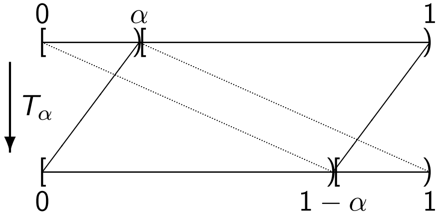

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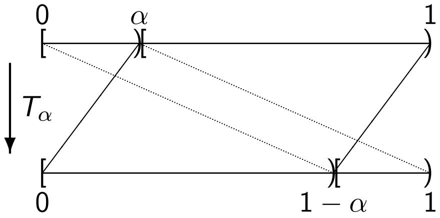

$L_N(\alpha)$ — set of factors of $t_{\alpha,\rho}$ of length N

Property

- α *irrational:* $\#L_N(\alpha) = C(t_{\alpha,\rho}, N) = N + 1$
- $p < N$, $p \perp N$: $\#L_N(p/N) = N$

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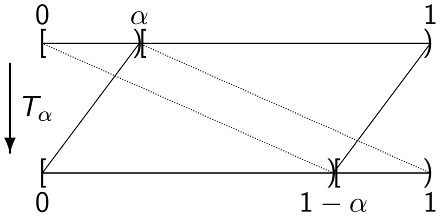

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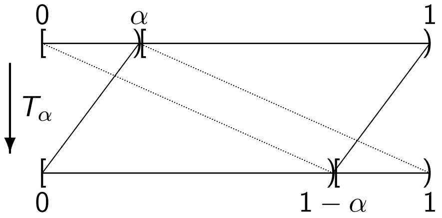

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Theorem

Let $p < N$, $p \perp N$, put $m = \min\{p, N-p\}$. Let $b \geq 0$. Then

$$\text{number of } b\text{-amicable pairs in } L_N(p/N) = \begin{cases} N - b & 0 \leq b \leq m \\ 0 & \text{otherwise} \end{cases}$$

Proof. Suppose $p \leq N/2$.

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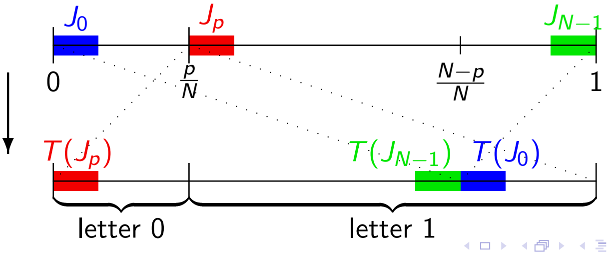


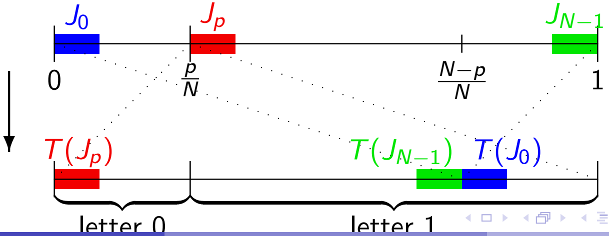
Proof of Theorem I.

Example ($L_8(3/8)$)

w_0	01011011
w_1	01101011
w_2	01101101
w_3	10101101
w_4	10110101
w_5	10110110
w_6	11010110
w_7	11011010

Observations:

- $w_i \overset{\text{lex}}{\leq} w_j \iff i \leq j$ hence
- $i > j \implies w_i \not\propto w_j$
- $J_i, T(J_i), T^2(J_i), \ldots, T^{N-1}(J_i)$ covers all J_j, $j = 0, \ldots, N-1$



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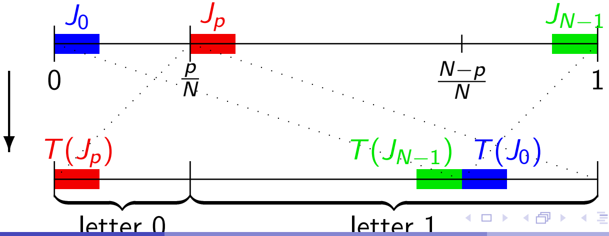

Proof of Theorem I.

Example ($L_8(3/8)$)

w_0	01011011
w_1	01101011
w_2	01101101
w_3	10101101
w_4	10110101
w_5	10110110
w_6	11010110
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Observations:

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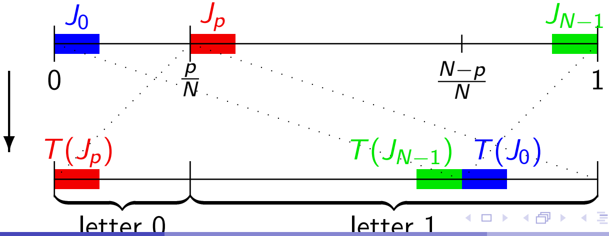

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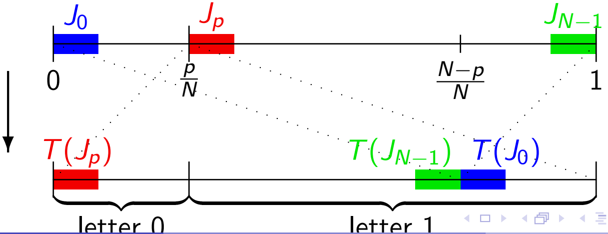

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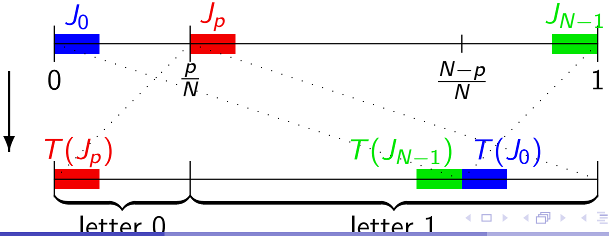

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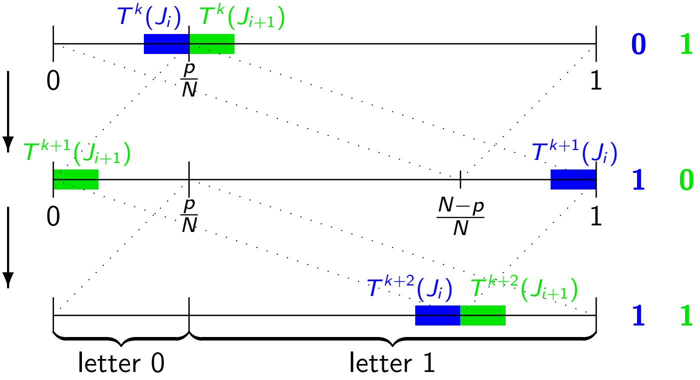

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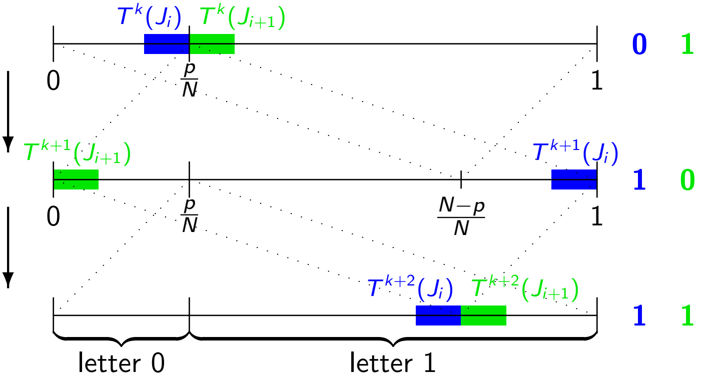

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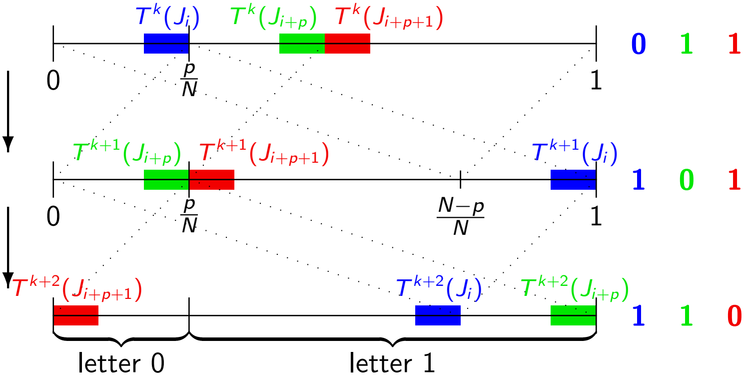
- for the case $j = i + p$ and $j = i + p + 1$



- $b \leq p \implies w_i$ is b-amicable to w_{i+b}
- $b \geq p + 1 \implies w_i \not\propto w_{i+b}$

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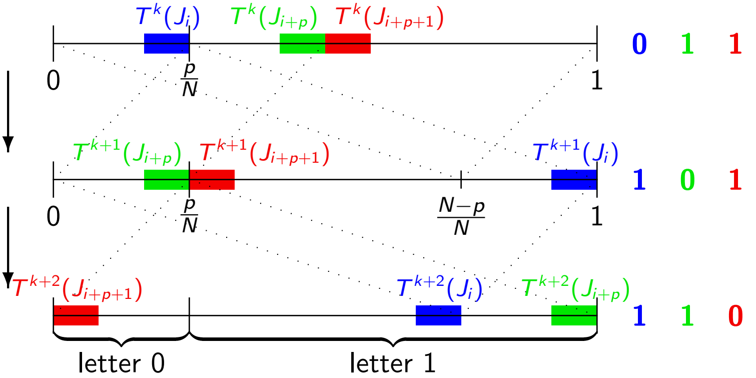

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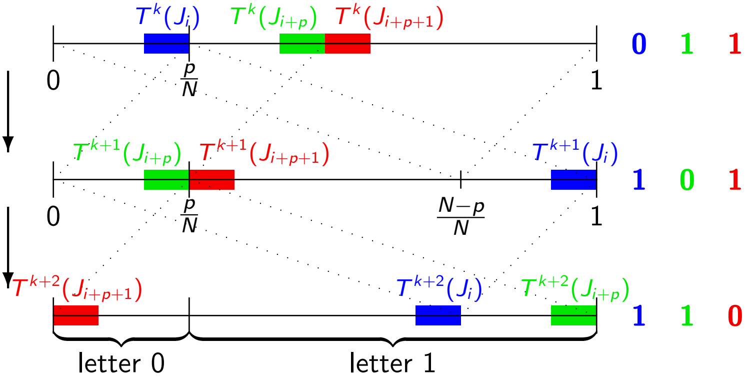

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Summary:

- $j < i$ ... w_i is not amicable to w_j
- $j = i$ ... w_i is 0-amicable to w_j ... N times
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$\vdots$ $\vdots$ $\vdots$

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For $p > N/2$: $p \longleftrightarrow N - p$ $0 \longleftrightarrow 1$

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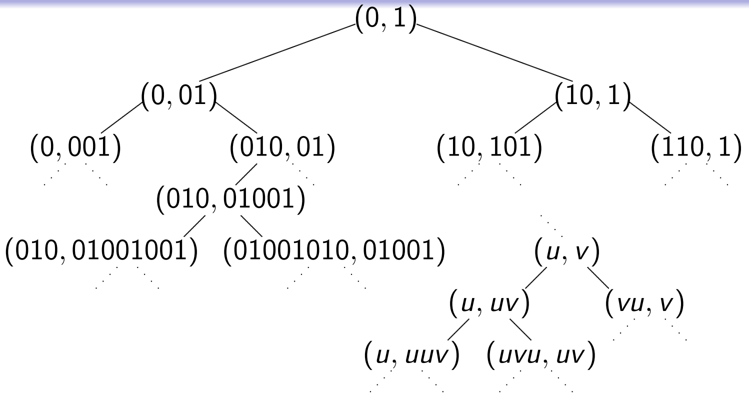
□

Standard Word Tree



Property

1. (u, v) *standard* $\xleftrightarrow{\text{one-to-one}}$ $A \in \mathbb{N}^{2,2}, \det A = 1$
2. $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{N}^{2,2}, \det A = 1 \implies a + b \perp c + d$

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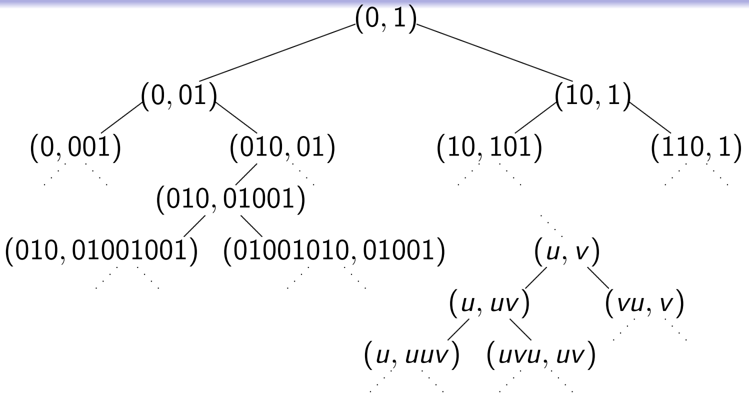

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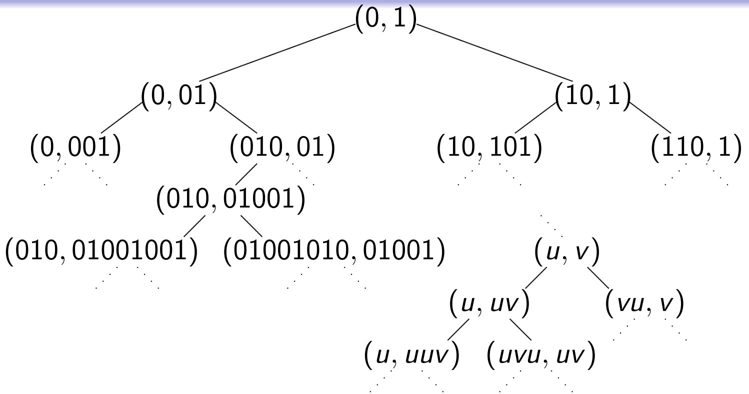

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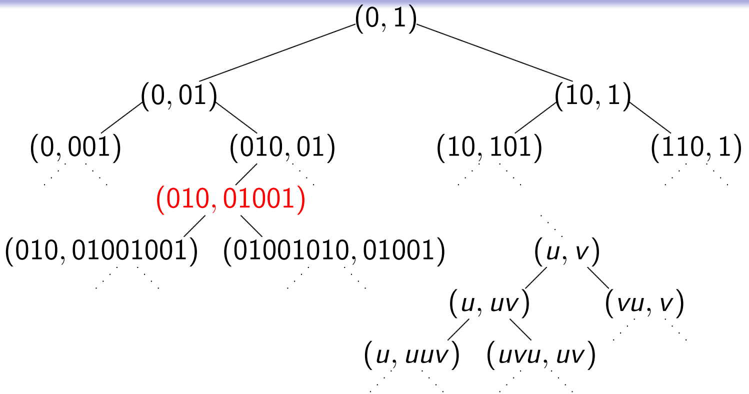

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Standard Morphisms → Sturmian Morphisms

Example

$\varphi^{[0]}$: 0→010, 1→01001 — standard
$\varphi^{[1]}$: 0→100, 1→10010
$\varphi^{[2]}$: 0→001, 1→00101
$\varphi^{[3]}$: 0→010, 1→01010 sturmian
$\varphi^{[4]}$: 0→100, 1→10100
$\varphi^{[5]}$: 0→001, 1→01001
$\varphi^{[6]}$: 0→010, 1→10010
$\varphi^{[7]}$: 0→101, 1→00100 — non-sturmian

$$\det M_{\varphi^{[k]}} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \quad k = 0, \ldots, 6$$

$$\det M_{\varphi^{[7]}} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \neq 1$$

Standard Morphisms → Sturmian Morphisms

Example

$\varphi^{[0]}$: $0 \to 010$, $1 \to 01001$ — standard
$\varphi^{[1]}$: $0 \to 100$, $1 \to 10010$
$\varphi^{[2]}$: $0 \to 001$, $1 \to 00101$
$\varphi^{[3]}$: $0 \to 010$, $1 \to 01010$
$\varphi^{[4]}$: $0 \to 100$, $1 \to 10100$
$\varphi^{[5]}$: $0 \to 001$, $1 \to 01001$
$\varphi^{[6]}$: $0 \to 010$, $1 \to 10010$

($\varphi^{[0]}$ – $\varphi^{[6]}$: sturmian)

$\varphi^{[7]}$: $0 \to 101$, $1 \to 00100$ — non-sturmian

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\varphi^{[3]}: 0\to010,\ 1\to01010 & \text{sturmian (}\varphi^{[0]}\text{–}\varphi^{[6]}\text{)} \\
\varphi^{[4]}: 0\to100,\ 1\to10100 & \\
\varphi^{[5]}: 0\to001,\ 1\to01001 & \\
\varphi^{[6]}: 0\to010,\ 1\to10010 & \\
\varphi^{[7]}: 0\to101,\ 1\to00100 & \text{— non-sturmian}
\end{array}$$

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Notation

- (u, v) standard pair, $N = |uv|$, $p = |uv|_0$
- $z^{[0]} = uv, \quad \tilde{z}^{[0]} = vu$
- $z^{[k]}, \tilde{z}^{[k]}$ k-th right conjugate $k = 0, \ldots, N-1$
- $z^{[k]} = u^{[k]} v^{[k]}, \quad \tilde{z}^{[k]} = v^{[k]} u^{[k]}$
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Theorem II.

Theorem

Let (u, v) *be a standard pair,* $N = |uv|$, $p = |uv|_0$, *put* $m = \min\{p, N-p\}$. *Let* $u^{[k]}$, $v^{[k]}$, $z^{[k]}$, $\tilde{z}^{[k]}$, $\varphi^{[k]}$, $\tilde{\varphi}^{[k]}$. *Let* $b \in \mathbb{N}$. *Then*

$$\#\text{ of b-amicable pairs in } \{\varphi^{[k]}\}_{k=0}^{N-2} = \begin{cases} N - b & 1 \le b \le m \\ 0 & \text{otherwise} \end{cases}$$

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Proof.

1. Proof for $\varphi^{[k]}$
2. Proof for $\tilde{\varphi}^{[k]}$

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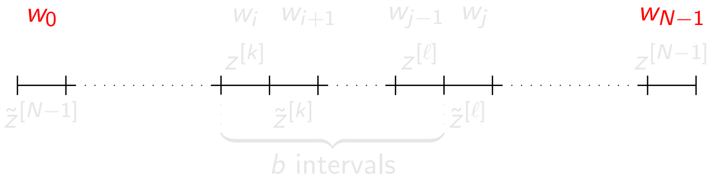
Proof of Theorem II. — For $\varphi^{[k]}$

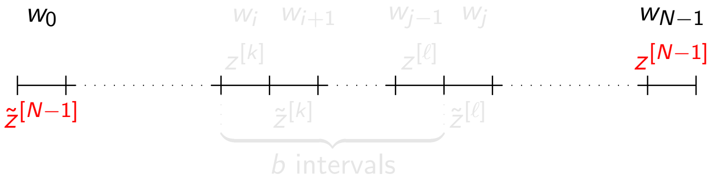


- For $k \neq N-1$, $\quad z^{[k]} = \varphi^{[k]}(01) \propto \varphi^{[k]}(10) = \tilde{z}^{[k]}$
- Take two morphisms $\varphi^{[k]}$, $\varphi^{[\ell]}$
- $b = j - i$, $0 \leq b \leq m$, $\quad z^{[k]}$ is b-amicable to $\tilde{z}^{[\ell]}$
- $z^{[k]}$ is $(b-1)$-amicable to $z^{[\ell]}$ and $\left|u^{[k]}\right|_0 = \left|u^{[\ell]}\right|_0$ hence

$$\varphi^{[k]}(0) = u^{[k]} \propto u^{[\ell]} = \varphi^{[\ell]}(0)$$

$$\varphi^{[k]}(1) = v^{[k]} \propto v^{[\ell]} = \varphi^{[\ell]}(1)$$

Proof of Theorem II. — For $\varphi^{[k]}$



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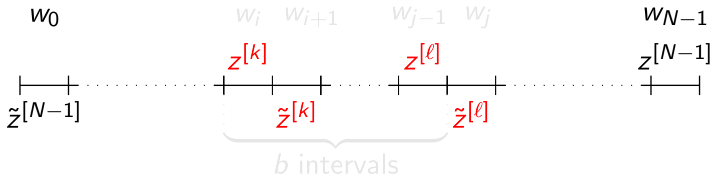
- Take two morphisms $\varphi^{[k]}$, $\varphi^{[\ell]}$
- $b = j - i$, $0 \leq b \leq m$, $\quad z^{[k]}$ is b-amicable to $\tilde{z}^{[\ell]}$
- $z^{[k]}$ is $(b-1)$-amicable to $z^{[\ell]}$ and $\left|u^{[k]}\right|_0 = \left|u^{[\ell]}\right|_0$ hence

$$\varphi^{[k]}(0) = u^{[k]} \propto u^{[\ell]} = \varphi^{[\ell]}(0)$$

$$\varphi^{[k]}(1) = v^{[k]} \propto v^{[\ell]} = \varphi^{[\ell]}(1)$$

Proof of Theorem II. — For $\varphi^{[k]}$



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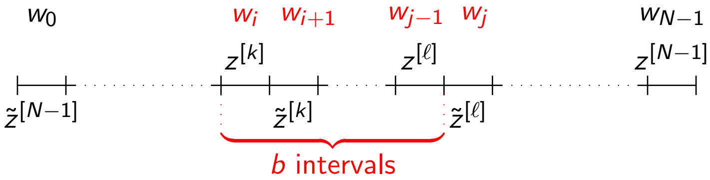

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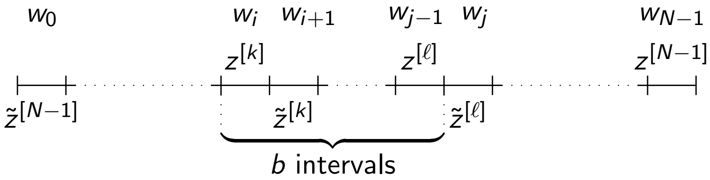

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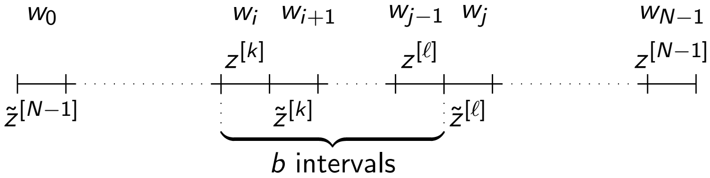
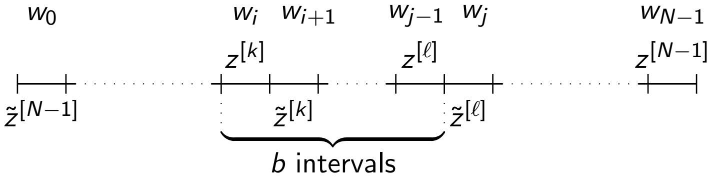


- $z^{[k]}$ is $(b-1)$-amicable to $z^{[\ell]}$ disallows $b = 0$, hence count is non-zero only for $1 \leq b \leq m$
- Number of $\varphi^{[k]}$ b-amicable to $\varphi^{[\ell]}$ is equal to number of (i, j), $0 \leq i, j \leq N - 1$ such that $j - i = b$, which is equal to $N - b$

Proof of Theorem II. — For $\varphi^{[k]}$



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Proof of Theorem II.

Theorem

Let (u, v) be a standard pair, $N = |uv|$, $p = |uv|_0$, put $m = \min\{p, N-p\}$. Let $u^{[k]}$, $v^{[k]}$, $z^{[k]}$, $\tilde{z}^{[k]}$, $\varphi^{[k]}$, $\tilde{\varphi}^{[k]}$. Let $b \in \mathbb{N}$. Then

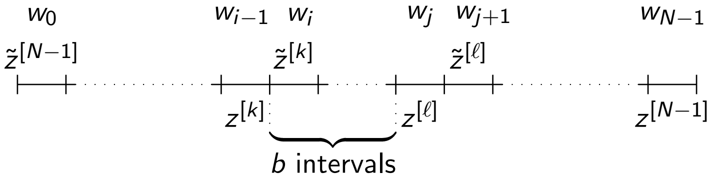
$$\#\text{ of b-amicable pairs is } \{\varphi^{[k]}\}_{k=0}^{N-2} = \begin{cases} N-b & 1 \leq b \leq m \\ 0 & \text{otherwise} \end{cases}$$

$$\#\text{ of b-amicable pairs is } \{\tilde{\varphi}^{[k]}\}_{k=0}^{N-2} = \begin{cases} N-b-2 & 0 \leq b \leq m-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof.

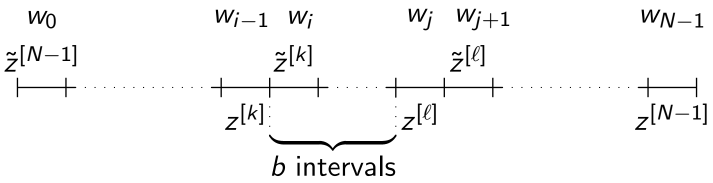
1. Proof for $\varphi^{[k]}$
2. Proof for $\tilde{\varphi}^{[k]}$

Proof of Theorem II. — For $\tilde{\varphi}^{[k]}$



- $j - i = b$, $\tilde{\varphi}^{[k]}(01) = \tilde{z}^{[k]}$ is b-amicable to $z^{[\ell]} = \tilde{\varphi}^{[\ell]}(10)$
- $k, \ell \neq N-1 \implies$ restrictions $i \geq 1$, $j \leq N-2$
- We need $z^{[k]} \propto z^{[\ell]}$, hence $j - (i-1) = b + 1 \leq m$
- Overall $0 \leq b \leq m-1$, (number of pairs) $= (N-2) - b$

Proof of Theorem II. — For $\tilde{\varphi}^{[k]}$



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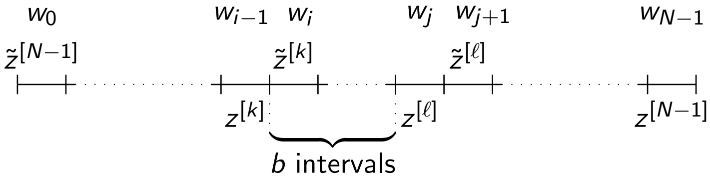

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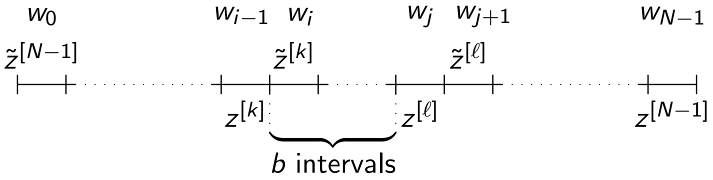

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□

Theorem III.

Theorem

Let $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in \mathbb{N}^{2,2}$, $\det A = \pm 1$. *Put* $p = a_{00} + a_{01}$, $N = a_{00} + a_{01} + a_{10} + a_{11}$, $m = \min\{p, N-p\}$. *Then number of amicable sturmian morphisms with matrix A is equal to*

$$m(N-1) + \frac{m}{2}(\det A - m)$$

Proof.

1. $\det A = +1$: $\{\varphi^{[k]}\}$ $\sum_{b=1}^{m} N - b$
2. $\det A = -1$: $\{\tilde{\varphi}^{[k]}\}$ $\sum_{b=0}^{m-1} N - 2 - b$

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