On the negative base greedy and lazy representations

Tomáš Hejda[†,*], Zuzana Masáková[†], Edita Pelantová[†]

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Abstract

We consider positional numeration systems with negative real base $-\beta$, where $\beta > 1$, and study the extremal representations in these systems, called here the greedy and lazy representations. We give algorithms for determination of minimal and maximal $(-\beta)$-representation with respect to the alternate order. We also show that both extremal representations can be obtained using the positive base β^2 and a non-integer alphabet. This enables us to characterize digit sequences admissible as greedy and lazy $(-\beta)$-representation. Such a characterization allows us to study the set of uniquely representable numbers. In case that β is the golden ratio, we give the characterization of digit sequences admissible as greedy and lazy $(-\beta)$-representation using a set of forbidden strings.

1 Introduction

For any real base α with $|\alpha| > 1$ and a finite alphabet of digits $\mathcal{A} \subset \mathbb{R}$, we consider a positional numeration system. If $x = \sum_{i \leqslant k} x_i \alpha^i$ with digits $x_i \in \mathcal{A}$, we say that x has an α-representation in $\mathcal{A}$. The assumption that every $x \in \mathbb{R}$ has a representation implies the condition on the cardinality of the alphabet $\#\mathcal{A} \geqslant |\alpha|$.

Rényi [8] showed that if $\alpha > 1$ and $\mathcal{A} = \{0, \dots \lceil \alpha \rceil - 1\}$, then every positive $x \geqslant 0$ has a representation. However, some numbers can have multiple representations. For example, in the decimal system one has

$$\tfrac{1}{2} = 0.5000000\ldots = 0.4999999\ldots, \qquad \text{whereas} \qquad \tfrac{1}{3} = 0.333333\ldots$$

If the base α is not an integer and the alphabet $\mathcal{A}$ is rich enough to represent all positive reals, then almost all $x \geqslant 0$ have infinitely many representations and one can choose among them "the nicest" one from some point of view.[1]

We consider the natural ordering of the alphabet and, for positive bases, the lexicographical ordering of the representations. Amongst all representations of a given $x \in \mathbb{R}$, we can choose the smallest and largest ones with respect to the lexicographical ordering; they are called the greedy and lazy representations. The study of the greedy representations started by Rényi. The study of the lazy representations was initialized by Erdős, Joó and Komornik [4] and they were extensively studied by Dajani and Kraaikamp [2].

The focus on the negative bases started by the work of Ito and Sadahiro in 2009 [5]. Representations in the negative base do not need the extra bit for the signum $\pm$. A family of transformations producing negative base representations of numbers for $1 < \beta < 2$ is studied in [1]. Among other, it is shown that although none of them gives the maximal representation

[*]Corresponding author, tohecz@gmail.com

[†]Doppler Institute for Mathematical Physics and Applied Mathematics and Department of Mathematics, FNSPE, Czech Technical University in Prague, supported by Grant Agency of the Czech Republic *GAČR 201/09/0584*, Ministry of Education, Youth and Sports of the Czech Republic *MSM6840770039*, Grant Agency of the Czech Technical University in Prague *SGS11/162/OHK4/3T/14*

[1]Note that although most people prefer writing $\frac{1}{2} = 0.5$, the shopkeepers consider the representation $0.4999\ldots$ nicer than $0.5000\ldots$ for $x = \frac{1}{2}$.

in the alternate order, it is produced by a random algorithm, see Theorem 4.2. in [1]. In its proof, one can find out that the greedy representation is obtained by periodic application of two transformations.

In this contribution, we focus on negative bases $-\beta$, $\beta > 1$ in general, and we deduce analogous result without introducing random $(-\beta)$-expansions. Our main result states that both extremal representations can be obtained using the positive base β^2 and a non-integer alphabet $\mathcal{B}$ by application of a transformation of the form $T(x) = \beta^2 x - D(x)$, where $D(x) \in \mathcal{B}$ (Theorem 4.1 and Proposition 4.2). Note that representations using a non-integer alphabet were considered by Pedicini [7]. This enables us to exploit results of [6] for giving necessary and sufficient conditions for identifying sequences admissible as greedy and lazy $(-\beta)$-expansions (Theorem 5.2). We give examples of the golden mean $\phi \approx 1.618$ and the tribonacci constant $\mu \approx 1.839$. In the last section, we discuss the uniqueness of the representations of numbers.

2 How to obtain α-representations of real numbers

In this chapter we recall a method for finding an α-representation of a given number with general real base α, $|\alpha| > 1$. It is clear that if we are able to find a representation for all numbers x from some bounded interval $J \subset \mathbb{R}$ containing 0, then we are also able to find an α-representation for any $x \in \bigcup_{k\in\mathbb{Z}} \alpha^k J$, i.e. for any real number (if 0 is inside J or the base α is negative) or for all positive reals or all negative reals (if 0 is a boundary point of J and the base α is positive). Our definition below is a restriction of the very general numeration system considered by Thurston [10], which permits for the base also complex numbers.

Definition 2.1. Given a base $\alpha \in \mathbb{R}$, $|\alpha| > 1$, a finite alphabet $\mathcal{A} \subset \mathbb{R}$ and a bounded interval $J \ni 0$. Let $D : J \mapsto \mathcal{A}$ be a mapping such that the transformation defined by $T(x) = \alpha x - D(x)$ maps $J \mapsto J$. The corresponding α-representation of x is a mapping $d = d_{\alpha,J,D} : J \mapsto \mathcal{A}^{\mathbb{N}}$,

$$d_{\alpha,J,D}(x) = x_1x_2x_3x_4\cdots, \quad \text{where } x_k = D\big(T^{k-1}(x)\big).$$

Both Rényi and Ito-Sadahiro systems [8, 5] are "order-preserving", provided that we choose a suitable order on the set of representations.

Definition 2.2. Let $\mathcal{A} \subset \mathbb{R}$ be a finite alphabet ordered by the natural order "$<$" in $\mathbb{R}$. Let $x_1x_2x_3\cdots$ and $y_1y_2y_3\cdots$ be two different strings from $\mathcal{A}^{\mathbb{N}}$. Denote $k = \min\{i \mid x_i \neq y_i\}$. We write

- $x_1x_2x_3\cdots \prec_{\text{lex}} y_1y_2y_3\cdots$ if $x_k < y_k$ and say that $x_1x_2x_3\cdots$ is smaller than $y_1y_2y_3\cdots$ in the lexicographical order;
- $x_1x_2x_3\cdots \prec_{\text{alt}} y_1y_2y_3\cdots$ if $(-1)^k x_k < (-1)^k y_k$ and say that $x_1x_2x_3\cdots$ is smaller than $y_1y_2y_3\cdots$ in the alternate order.

Proposition 2.3. *Let α, $\mathcal{A}$, J and D be as in Definition 2.1. Let numbers $x, y \in J$ and let $d_{\alpha,J,D}(x) = x_1x_2x_3\cdots$ and $d_{\alpha,J,D}(y) = y_1y_2y_3\cdots$ be their α-representations.*

- *If $\alpha > 1$ and $D(x)$ is non-decreasing then*
$$x < y \iff x_1x_2x_3\cdots \prec_{\text{lex}} y_1y_2y_3\cdots.$$
- *If $\alpha < -1$ and $D(x)$ is non-increasing then*
$$x < y \iff x_1x_2x_3\cdots \prec_{\text{alt}} y_1y_2y_3\cdots.$$

For a given base α and an alphabet $\mathcal{A} \subset \mathbb{R}$ we put

$$J_{\alpha,\mathcal{A}} = \left\{ \sum_{i=1}^{\infty} a_i\alpha^{-i} \,\middle|\, a_i \in \mathcal{A} \right\},$$

the set of numbers representable with negative powers of α and the alphabet $\mathcal{A}$, and for any $x \in J_{\alpha,\mathcal{A}}$, we denote the set of its α-representations in $\mathcal{A}$ by

$$R_{\alpha,\mathcal{A}}(x) = \big\{x_1x_2x_3\cdots \,\big|\, x = \textstyle\sum_{i=1}^{\infty} x_i\alpha^{-i} \text{ and } x_i \in \mathcal{A}\big\}.$$

The Proposition 2.3 suggests how to choose a suitable ordering on the set $R_{\alpha,\mathcal{A}}(x)$.

Definition 2.4. Let α be a real base with $|\alpha| > 1$ and let $\mathcal{A} \subset \mathbb{R}$ be a finite alphabet.

- Let $\alpha < -1$ and $x \in J_{\alpha,\mathcal{A}}$. Then the maximal and minimal elements of $R_{\alpha,\mathcal{A}}(x)$ with respect to the alternate order are called the greedy and lazy α-representations of x in the alphabet $\mathcal{A}$, respectively.
- Let $\alpha > 1$ and $x \in J_{\alpha,\mathcal{A}}$. Then the maximal and minimal elements of $R_{\alpha,\mathcal{A}}(x)$ with respect to the lexicographical order are called the greedy and lazy α-representations of x in the alphabet $\mathcal{A}$, respectively.

3 Extremal representations in negative base systems

Let us now fix a base $\alpha = -\beta$ for some non-integer $\beta > 1$, $\beta \notin \mathbb{N}$, and an alphabet $\mathcal{A} = \{0, 1, \dots, \lfloor\beta\rfloor\}$. Using the same arguments as in [7] it can be shown that the set I of numbers representable in this system is an interval, namely

$$I = \Big[\frac{-\beta\lfloor\beta\rfloor}{\beta^2-1}, \frac{\lfloor\beta\rfloor}{\beta^2-1}\Big] =: [l, r]. \tag{1}$$

The interval I (i.e. its boundary points) depend on the base β, however, we avoid it in the notation for simplicity. We denote by I_a the set of numbers which have a $(-\beta)$-representation starting with the digit $a \in \mathcal{A}$. Then $I_a = \frac{a}{-\beta} + \frac{1}{-\beta}I = \Big[\frac{a}{-\beta} + \frac{r}{-\beta}, \frac{a}{-\beta} + \frac{l}{-\beta}\Big]$ and I can be written as a (not necessarily disjoint) union of intervals $I = \bigcup_{a\in\mathcal{A}} I_a$. Obviously, we have $-\beta x - a \in I$ for every $x \in I_a$. Note that intervals I_a overlap, but not three at a time.

We define

$$D_m(x) = \begin{cases} \lfloor\beta\rfloor & \text{for } x \in I_{\lfloor\beta\rfloor}, \\ a & \text{for } x \in I_a \setminus I_{a+1},\ a \in \mathcal{A},\ a \neq \lfloor\beta\rfloor. \end{cases}$$

and

$$D_v(x) = \begin{cases} 0 & \text{for } x \in I_0, \\ a & \text{for } x \in I_a \setminus I_{a-1},\ a \in \mathcal{A},\ a \neq 0. \end{cases}$$

and corresponding transformations

$$T_m(x) = -\beta x - D_m(x) \qquad \text{and} \qquad T_v(x) = -\beta x - D_v(x)\,.$$

Proposition 3.1. *Let $x \in I$.*

- *Denote $\varepsilon_0 = x$ and for all $i \geqslant 0$ put*
  $$z_{2i+1} = D_m(\varepsilon_{2i}),\ \varepsilon_{2i+1} = T_m(\varepsilon_{2i}) \quad \text{and} \quad z_{2i+2} = D_v(\varepsilon_{2i+1}),\ \varepsilon_{2i+2} = T_v(\varepsilon_{2i+1}).$$
  Then $z_1z_2z_3\cdots$ is the greedy $(-\beta)$-representation of x.
- *Denote $\eta_0 = x$ and for all $i \geqslant 0$ put*
  $$y_{2i+1} = D_v(\eta_{2i}),\ \eta_{2i+1} = T_v(\eta_{2i}) \quad \text{and} \quad y_{2i+2} = D_m(\eta_{2i+1}),\ \eta_{2i+2} = T_m(\eta_{2i+1}).$$
  Then $y_1y_2y_3\cdots$ is the lazy $(-\beta)$-representation of x.

For $\beta \in (1, 2)$, the description of greedy $(-\beta)$-representations of Proposition 3.1 can be deduced from the proof of Theorem 4.2 in [1], albeit the statement of the theorem is vague, namely that the greedy $(-\beta)$-representation can be generated by a "random sequence of transformations".

4 Representations in base β^2 with non-integer alphabets

The algorithm for obtaining extremal $(-\beta)$-representations of a number x, described in Proposition 3.1, does not fit in the scheme of the Definition 2.1 for negative base $\alpha = -\beta$. In particular, there is no transformation $T(x) = -\beta x - D(x)$ which generates for every x the greedy (or lazy) $(-\beta)$-representation. This fact complicates the description of digit strings occurring as greedy or lazy representations. Nevertheless, we arrive to overcome this handicap.

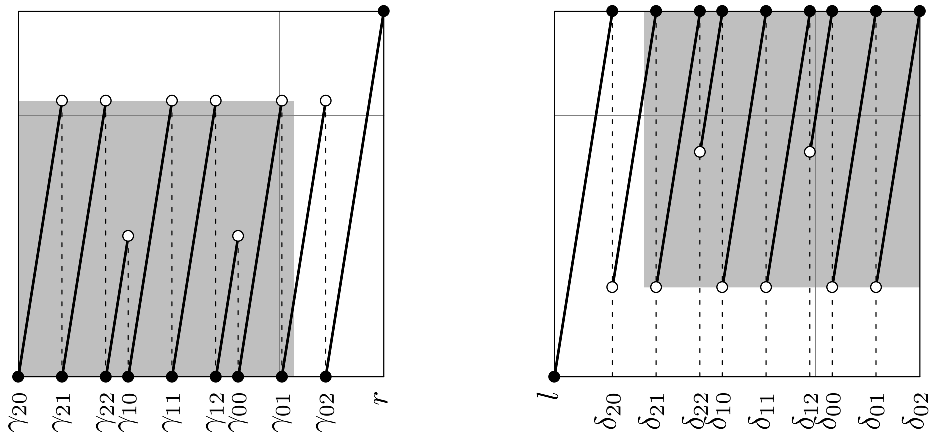

Figure 1: The greedy and lazy transformations T_G and T_L. This figure corresponds to a base $-\beta \in (-3,-2)$. Here, $\delta_{ba} = \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{r}{(-\beta)^2}$.

Defining $T_G := T_m T_v$ and $T_L := T_v T_m$ we obtain transformations $I \to I$ which produce the greedy and lazy $(-\beta)$-representations. The price to be paid is that the digit assigning functions D_G and D_L are not integer-valued. Precisely, $D_G, D_L : I \to \mathcal{B} = \{-b\beta + a \mid a, b \in \mathcal{A}\}$. This alphabet $\mathcal{B}$ has $(\#\mathcal{A})^2$ distinct elements, since we consider only $\beta \notin \mathbb{N}$.

Let us describe the mappings T_G and D_G (resp. T_L and D_L) explicitly. Put

$$\gamma_{ba} = \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{l}{(-\beta)^2} \quad \text{for any } a, b \in \mathcal{A}$$

$$\text{and} \quad D_G(x) = \max\big\{-b\beta + a \,\big|\, a, b \in \mathcal{A} \text{ and } \gamma_{ba} \leqslant x\big\}. \tag{2}$$

Notice that the set in the definition of D_G is non-empty since $\gamma_{\lfloor\beta\rfloor 0} = l \leqslant x$ for all $x \in I$.

Defining a morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ by

$$\psi(-b\beta + a) = ba,$$

we can state the following theorem.

Theorem 4.1. *Let $\beta > 1$, $\beta \notin \mathbb{N}$, $\mathcal{A} = \{0, 1, \dots, \lfloor\beta\rfloor\}$. Define on the interval I from* (1) *the transformation $T_G : I \to I$ by the prescription*

$$T_G(x) = \beta^2 x - D_G(x)\,,$$

where $D_G : I \to \mathcal{B}$ is given by (2). *For an $x \in I$ denote by $d_G(x)$ the corresponding β^2-representation of x. Then*

- *$d_G(x)$ is the greedy β^2-representation of x in the alphabet $\mathcal{B}$.*
- *$\psi\big(d_G(x)\big)$ is the greedy $(-\beta)$-representation of x in the alphabet $\mathcal{A}$.*

Using the following generalization of result in [4], we can obtain the lazy $(-\beta)$-representations of numbers from the greedy ones, and in the following text, we will mainly concentrate on the properties of the greedy transformation T_G.

Proposition 4.2. *Let $z_1 z_2 z_3 z_4 \cdots$ be the greedy $(-\beta)$-representation of a number $z \in I$ and let $y_1 y_2 y_3 y_4 \cdots$ be the lazy $(-\beta)$-representation of a number $y \in I$. Then*

$$y_i + z_i = \lfloor\beta\rfloor \quad \textit{for every } i \geqslant 1 \qquad \Longleftrightarrow \qquad y + z = -\frac{\lfloor\beta\rfloor}{\beta+1}\,.$$

For an example of transformations T_G, T_L, see Figure 1.

5 Admissibility

The transformation T_G has the following property.

Lemma 5.1. *For every $x \in [l, l+1)$ one has $T_G(x) \in [l, l+1)$. Moreover, for every $x \in I \setminus [l, l+1)$, $x \neq r$, there exists an exponent $k \in \mathbb{N}$ such that $T_G^k(x) \in [l, l+1)$.*

The above fact implies that, in general, some digits from the alphabet $\mathcal{B}$ do not appear infinitely many times in the greedy β^2-representation of any number $x \in I$. Let $\mathcal{A}_G \subseteq \mathcal{B}$ comprise those digits that can appear infinitely many times, i.e. $\mathcal{A}_G = \{D_G(x) \,|\, x \in [l, l+1)\}$, analogously we put $\mathcal{A}_L = \{D_L \,|\, x \in (r-1, r]\}$.

In order to formulate the result about admissible greedy representations which is derived using a result of [6], we introduce the left-continuous mappings $D_G^* : I \to \mathcal{B}$, $T_G^* : I \to I$ and $d_G^* : I \to \mathcal{A}^{\mathbb{N}}$ as

$$D_G^*(x) = \lim_{\varepsilon \to 0+} D_G(x - \varepsilon)\,, \quad T_G^*(x) = \lim_{\varepsilon \to 0+} T_G(x - \varepsilon)\,, \quad d_G^*(x) = \lim_{\varepsilon \to 0+} d_G(x - \varepsilon)\,.$$

Theorem 5.2. *Let $X_1 X_2 X_3 \cdots \in \mathcal{A}_G^{\mathbb{N}}$. Then there exists an $x \in [l, l+1)$ such that $d_G(x) = X_1 X_2 X_3 \cdots$ if and only if for every $k \geqslant 1$*

$$X_{k+1} X_{k+2} X_{k+3} \cdots \prec \begin{cases} d_G^*\big(T_G^*(l+1)\big) & \text{if } X_k = \max \mathcal{A}_G, \\ d_G^*\big(l + \{\beta\}\big) & \text{if } X_k = -b\beta + \lfloor \beta \rfloor,\ X_k \neq \max \mathcal{A}_G. \end{cases}$$

Remark 5.3. Using Proposition 4.2 one can derive an analogous necessary and sufficient condition for admissible lazy β^2-representations $X_1 X_2 X_3 \cdots$ of numbers in $x \in (r-1, r]$ over the alphabet $\mathcal{A}_L = \lfloor \beta \rfloor - \mathcal{A}_G$.

6 Negative golden ratio

Let us illustrate the previous results and their implications on the example of the negative base $-\beta$ where $\beta = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. Real numbers representable in base $-\phi$ over the alphabet $\mathcal{A} = \{0, 1\}$ form the interval $J_{-\phi, \mathcal{A}} = I = \big[-1, \frac{1}{\phi}\big] = [l, r]$.

The greedy and lazy $(-\phi)$-representation can be obtained from the greedy and lazy ϕ^2-representation over the alphabet $\mathcal{B} = \{-\phi, -\phi+1, 0, 1\}$, applying the morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ given by

$$\psi(-\phi) = 10, \quad \psi(-\phi+1) = 11, \quad \psi(0) = 00, \quad \psi(1) = 01.$$

The greedy and lazy ϕ^2-representations are generated by the transformation

$$T_G(x) = \phi^2 x - D_G(x)\,, \quad T_L(x) = \phi^2 x - D_L(x)\,, \quad x \in \big[-1, \tfrac{1}{\phi}\big]\,,$$

where the digit assigning maps D_G and D_L are

$$D_G(x) = \begin{cases} -\phi & \text{for } x \in \big[-1, -\frac{1}{\phi}\big), \\ -\phi+1 & \text{for } x \in \big[-\frac{1}{\phi}, -\frac{1}{\phi^2}\big), \\ 0 & \text{for } x \in \big[-\frac{1}{\phi^2}, 0\big), \\ 1 & \text{for } x \in \big[0, \frac{1}{\phi}\big], \end{cases} \qquad D_L(x) = \begin{cases} -\phi & \text{for } x \in \big[-1, -\frac{1}{\phi^2}\big], \\ -\phi+1 & \text{for } x \in \big(-\frac{1}{\phi^2}, 0\big], \\ 0 & \text{for } x \in \big(0, \frac{1}{\phi^3}\big], \\ 1 & \text{for } x \in \big(\frac{1}{\phi^3}, \frac{1}{\phi}\big]. \end{cases}$$

The graph of the transformations T_G, T_L restricted to the intervals $[l, l+1) = [-1, 0)$, $(r-1, r] = (\frac{1}{\phi^2}, -\frac{1}{\phi}]$ are drawn in Figure 2.

Let us now apply Theorem 5.2 to the case $\beta = \phi$. Denote for simplicity the digits of the alphabet $\mathcal{B} = \{-\phi, -\phi+1, 0, 1\}$ by

$$A = -\phi \quad < \quad B = -\phi + 1 \quad < \quad C = 0 \quad < \quad D = 1.$$

With this notation, we have $\mathcal{A}_G = \{A, B, C\}$ and $\mathcal{A}_L = \{B, C, D\}$.

Proposition 6.1. *A string $X_1 X_2 X_3 \cdots$ over the alphabet $\mathcal{A}_G = \{A, B, C\}$ is the greedy ϕ^2-representation of a number $x \in [-1, 0)$ if and only if it does not contain a factor from the set $\{BC, B^\omega, C^\omega\}$.*

A string $X_1 X_2 X_3 \cdots$ over the alphabet $\mathcal{A}_L = \{B, C, D\}$ is the lazy ϕ^2-representation of a number $x \in (-\frac{1}{\phi^2}, \frac{1}{\phi}]$ if and only if it does not contain a factor from the set $\{CB, B^\omega, C^\omega\}$.

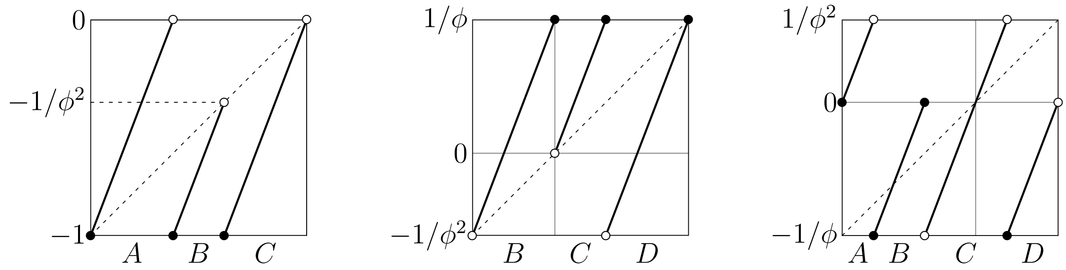

Figure 2: Transformations T_G (left), T_L (middle) and T_{IS}^2 (right) in the base ϕ^2 that correspond to the greedy, lazy and Ito-Sadahiro representations in the base $-\phi$.

Corollary 6.2. *The points $x = -1$ and $x = \frac{1}{\phi}$ are the only points in $[-1, \frac{1}{\phi}]$ which have a unique $(-\phi)$-representation over the alphabet $\{0, 1\}$.*

Proposition 6.1 provides a combinatorial criterion for admissibility of representations in base ϕ^2 in the non-integer alphabet $\mathcal{A}_G$. One can also rewrite the admissibility of a digit string in base $-\phi$ using forbidden strings in the original alphabet $\{0, 1\}$.

Proposition 6.3. *A digit string $x_1x_2x_3 \cdots$ over the alphabet $\{0, 1\}$ is a greedy $(-\phi)$-representation of some $x \in [-1, 0)$ if and only if*

(i) it does not start with the prefix $1^{2k}0$, nor $0^{2k-1}1$, $k \geqslant 1$;

(ii) it does not end with the suffix 0^ω nor 1^ω;

(iii) it does not contain the factor $10^{2k}1$, nor $01^{2k}0$, $k \geqslant 1$.

Corollary 6.4. *The Ito-Sadahiro $(-\phi)$-representation introduced in [5] is not extremal for any $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$.*

For the plots of the greedy, lazy and Ito-Sadahiro representations, see Figure 2.

7 Unique $(-\beta)$-representations

In [1], it is shown that for $1 < \beta < 2$, the set of numbers with a unique $(-\beta)$-representation is of Lebesgue measure zero. The authors also show that for $\beta < \phi$, such numbers are only two. Let us show that although the measure is always zero, the set of numbers with unique $(-\beta)$-representation can be uncountable.

Proposition 7.1. *Let μ be the Tribonacci constant, i.e. the real root $\mu \approx 1.839$ of $x^3 - x^2 - x - 1$. Denote $A = -\mu$, $B = -\mu + 1$, $C = 0$, $D = 1$ the alphabet $\mathcal{B}$ for this particular case. Then all strings over the letters $\{B, C\}$ are admissible as both greedy and lazy $(-\mu)$-representations.*

We can prove an analogous statement for all sufficiently large bases.

Theorem 7.2. *Let $\beta > 1 + \sqrt{3} \approx 2.732$, $\beta \notin \mathbb{N}$. Then there exist uncountably many numbers in $J_{-\beta,\mathcal{A}}$ having a unique $(-\beta)$-representation over the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$.*

8 Conclusions

Our main tool in this paper was to view the $(-\beta)$-representations in the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ as strings of pairs of digits in $\mathcal{A}$, which amounts, in fact, to considering the alphabet $\mathcal{B} = -\beta \cdot \mathcal{A} + \mathcal{A}$ and the base β^2. Such an approach puts forward the utility of studying number systems with positive base and a non-integer alphabet, as was already started by Pedicini [7] or Kalle and Steiner [6]. Obtaining new results for such systems – for example analogous to those of de Vries and Komornik [3] or Schmidt [9] would probably contribute also to the knowledge about negative base systems.

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