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Geometrical aspects of positional representations of real and complex numbers

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THESIS

Geometrical aspects of positional representations of real and complex numbers

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Geometrical aspects of positional representations of real and complex numbers.

Abstract. We study three geometrical aspects of positional numeration systems.

First, we study Rauzy fractals associated to the symmetric β-expansions. We recall that the symmetric β-expansion of $x \in [-\frac{1}{2}, \frac{1}{2})$ is the coding of its orbits by $x \mapsto \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$ with the code $x \mapsto \lfloor \beta x + \frac{1}{2} \rfloor$. For arbitrary Pisot unit β, the collection of the Rauzy fractals always forms a multiple tiling of the contracting hyperplane. We concentrate on the case $\beta \in (1, 2)$. We give a necessary condition on the coefficients of the minimal polynomial of β, under which the collection forms a single tiling. When this necessary condition holds, we reduce the tiling problem to a tiling problem for the Rauzy fractals of another type of β-expansions.

In the second part, we consider the greedy β-expansions for arbitrary quadratic Pisot number β. We recall that the greedy β-expansion of $x \in [0, 1)$ is the coding of its orbit by the map $x \mapsto \beta x - \lfloor \beta x \rfloor$ with the code $x \mapsto \lfloor \beta x \rfloor$. We are interested in the study of rational numbers that have a purely periodic expansion. For many values of β, there exists $c > 0$ such that all rational numbers in $[0, c)$ whose denominator is relatively prime to $\beta\beta'$ have a purely periodic expansion. We give an algorithm that determinines whether such $c > 0$ exists, and when it exists, this algorithm computes the maximum value of c with arbitrary precision. For cases when $\beta + \beta'$ is an integer multiple of $\beta\beta'$, we give a necessary and sufficient condition on $\max c = 1$.

The third part is devoted to the study of complex spectra. For a complex number $\gamma \in \mathbb{C} \setminus \mathbb{R}$ with $|\gamma| > 1$ and a finite alphabet $\mathcal{A}$ containing 0, we define the spectrum $\mathcal{A}[\gamma]$ as the set of all polynomials with coefficients in $\mathcal{A}$ evaluated at the point γ. We prove that when the alphabet is too small, namely when $\#\mathcal{A} < |\gamma|^2$, the spectrum is not relatively dense in $\mathbb{C}$. For a class of cubic complex Pisot units, we give an algorithm that determines the shortest distance between points of the spectra for all alphabets of the form $\mathcal{A} = \{0, 1, \ldots, m\}$ at once.

Aspects géométriques des représentations positionelles des nombres réels et complexes.

Résumé. Nous étudions trois aspects des systèmes de numération positionnels.

Nous commmençons par l'étude des fractals de Rauzy de β-développements symmétriques, c'est-à-dire, les codage de l'orbite de $x \in [-\frac{1}{2}, \frac{1}{2})$ par $x \mapsto \beta - \lfloor \beta x + \frac{1}{2} \rfloor$ avec le code $x \mapsto \lfloor \beta x + \frac{1}{2} \rfloor$. Pour chaque unité de Pisot β, les fractals de Rauzy forment un pavage multiple de l'hyperplan contractant. Nous considérons le cas $\beta \in (1, 2)$. Nous donnons une condition nécessaire pour que le pavage multiple soit un pavage. Sous cette condition, nous réduisons la question de pavage à une question de pavage pour des β-développements d'un autre type.

La deuxième partie concerne les nombres rationnels dont le développement de Rényi est purement périodique. Le développement de Rényi de $x \in [0, 1)$ est le codage de son orbite par la fonction $x \mapsto \beta x - \lfloor \beta x \rfloor$ avec le code $x \mapsto \lfloor \beta x \rfloor$. Pour beaucoup de nombres de Pisot quadratiques il existe $c > 0$ tel que le développement de Rényi de x est purement periodique pour tout $p/q \in [0, c)$ avec $p, q \in \mathbb{Z}$ et q premier avec $\beta\beta'$. Nous présentons un algorithme qui décide si un tel c existe; dans ce cas, il calcule la valeur maximale de c. Quand $\beta + \beta'$ est un multiple de $\beta\beta'$, nous trouvons une condition nécessaire et suffisante pour que $\max c = 1$.

La troisième partie est consacrée à l'etude des spectres de nombres complexes. Pour un nombre complexe $\gamma \in \mathbb{C} \setminus \mathbb{R}$ tel que $|\gamma| > 1$ et pour un alphabet fini contenant 0, nous définissons le spectre $\mathcal{A}[\gamma]$ comme l'ensemble de tous les polynômes dont les coefficients se trouvent dans $\mathcal{A}$, évalué en γ. Nous montrons que si l'alphabet est trop petit, c'est-à-dire, $\#\mathcal{A} < |\gamma|^2$, alors le spectre n'est pas relativement dense dans C. Pour une classe des unités cubiques complexes de Pisot, nous présentons un algorithm qui calcule la distance minimale entre les point des spectres pour tous les alphabets $\mathcal{A} = \{0, 1, \cdots, m\}$.

Geometrické pohledy na poziční reprezentace reálných a komplexních čísel.

Abstrakt. Zabýváme se třemi geometrickými hledisky pozičních numeračních soustav.

V první části studujeme racionální čísla, která mají čistě periodické hladové β-rozvoje, pro kvadratické Pisotova čísla β. Hladovým β-rozvojem čísla $x \in [0, 1)$ rozumíme kódování jeho orbity při transformaci $x \mapsto \beta x - \lfloor \beta x \rfloor$ pomocí kódu $x \mapsto \lfloor \beta x \rfloor$. Pro mnoho takových β existuje $c > 0$ takové, že všechna racionální čísla v intervalu $[0, 1)$, jejichž jmenovatel je nesoudělný s $\beta\beta'$, mají čistě periodický rozvoj. Navrhujeme algoritmus, který zjistí, zda takové c existuje, a pokud ano, určí maximální možnou hodnotu c s libovolnou přesností. V případě, kdy $\beta\beta'$ celočíselně dělí $\beta + \beta'$, představujeme nutnou a postačující podmínku, aby $\max c = 1$.

V druhé části se zabýváme studiem Rauzyho fraktálů pro symetrické β-rozvoje, tedy pro kódování orbit zobrazení $x \mapsto \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$. Pro libovolnou Pisotovu jednotku tvoří Rauzyho fraktály multidláždění příslušného prostoru. Pro $\beta \in (1, 2)$ ukazujeme nutnou podmínku na koeficienty minimálního polynomu β, aby toto multidláždění mohlo být dlážděním. Je-li tato nutná podmínka splněna, převádíme tuto otázku na problém spojený s Rauzyho fraktály pro jinou β-transformaci.

Třetí část je zaměřena na spektra komplexních čísel. Spektrum, pro $\gamma \in \mathbb{C} \setminus \mathbb{R}$, $|\gamma| > 1$ a pro konečnou abecedu $\mathcal{A}$ obsahující 0, je množina hodnot všech polynomů s koeficienty v $\mathcal{A}$ vyčíslených v bodě γ. Ukazujeme, že pro malé abecedy, $\#\mathcal{A} < |\gamma|^2$, spektrum není relativně husté v $\mathbb{C}$. Pro třídu kubických komplexních jednotek navrhujeme algoritmus, který najde nejmenší vzdálenost mezi prvky spektra s abecedou $\mathcal{A} = \{0, 1, \ldots, m\}$, a to pro všechna m najednou.

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Declaration

I hereby declare that this dissertation is my own original work and that I acknowledged all used references.

In Prague, 1st of January 2016

Tomáš Hejda

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CHAPTER ONE

Introduction

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Positional representations of numbers provide a very genuine way of expressing numbers as *words* — sequences of symbols, where only finitely many well-distinguishable different symbols appear. This genuinity has been recognized by many important mathematicians, such as Pierre-Simon Laplace, who said (according to [**Eve88**]):

> *"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."*

In the spirit of this quotation, we will consider positional representation in the form

$$x = \sum x_j b_j, \quad \text{written as } x = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} x_{-3} \cdots, \tag{1-1}$$

where x_j are from a finite alphabet of digits, and (b_j) is a sequence of numbers that is increasing in modulus.

The most classic example known to today's world is the decimal system, where $b_j \coloneqq 10^j$ and $x_j \in \mathcal{A}_9 \coloneqq \{0,1,2,3,4,5,6,7,8,9\}$. We can take any $\beta \in \mathbb{N}$ at least 2 and $x_j \in \mathcal{A}_{\beta-1} \coloneqq \{0,1,\dots,\beta-1\}$. For instance in the computer arithmetic, it is standard to work in the binary system $\beta = 2$.

If $x \in \mathbb{N}$, it can be written uniquely as $x = x_k x_{k-1} \cdots x_1 x_0 \bullet$ in the above system (uniquely up to leading zeros), and there are two well-known algorithms that compute the representation; the first one starts by computing x_0, the other one by computing x_k:

Greedy algorithm.

- Input: a number $x \in \mathbb{N}$ and a base $\beta \in \mathbb{N}$, $\beta \geqslant 2$.
- Output: $x_k \cdots x_1 x_0{}_\bullet$, the representation of x.

1. Find the largest k such that $\beta^k < x$, set $j \coloneqq k$.
2. Compute $x_j \coloneqq \lfloor x/\beta^j \rfloor$.
3. Set $x \coloneqq x - x_j\beta^j$ and $j \coloneqq j - 1$.
4. Repeat steps 2–3 as long as $j \geqslant 0$.

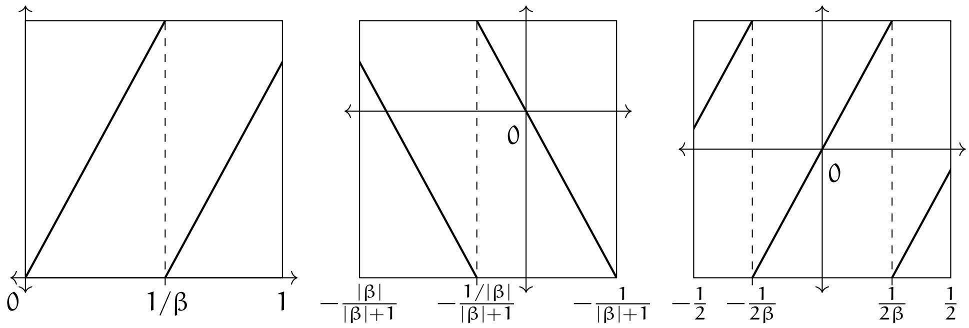
Division algorithm.

- Input: a number $x \in \mathbb{N}$ and a base $\beta \in \mathbb{N}$, $\beta \geqslant 2$.
- Output: $x_k \cdots x_1 x_0{}_\bullet$, the representation of x.

1. Set $j \coloneqq 0$.
2. Find $x_j \in \{0, 1, \dots, \beta - 1\}$ such that $x_j \equiv x \pmod{\beta}$.
3. Set $x \coloneqq (x - x_j)/\beta$ and $j \coloneqq j + 1$.
4. Repeat steps 2–3 until $x = 0$.

The greedy algorithm can be run on any real $x > 0$ to obtain a representation. A. Rényi [**Rén57**] then observed that it works for arbitrary real $\beta > 1$. The output of this algorithm for $x > 0$ is called the ***greedy* β*-expansion*** of x. K. Schmidt [**Sch80**] showed that all numbers from the field $\mathbb{Q}(\beta)$ have an eventually periodic β-expansion if β is a *Pisot number*, i.e., if $\beta > 1$ is an algebraic integer and all its other Galois conjugates are in modulus < 1. C. Frougny and B. Solomyak [**FS92**] then discussed what numbers have a finite β-expansion. Certainly if $x > 0$ has a finite β-expansion and β is an algebraic number then $x \in \mathbb{Z}[\beta, \beta^{-1}]$. They say that β has *Property (F)* if every $x \in \mathbb{Z}[\beta^{-1}] \cap [0, \infty)$ has a finite β-expansion. They also show that if the minimal polynomial of $\beta > 1$ is $P_\beta(X) = X^d - a_{d-1}X^{d-1} - \cdots - a_1X - a_0$ with $a_{d-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 \geqslant 1$, then β is a Pisot number and has Property (F). M. Hollander [**Hol96**] then showed that also if $a_j \geqslant 0$ and $a_{d-1} > a_0 + \cdots + a_{d-2}$ then β is a Pisot number with Property (F). S. Akiyama [**Aki00**] found all cubic Pisot units that satisfy Property (F). However, the problem of describing all numbers with Property (F) is considered difficult.

The division algorithm was generalized to non-integer bases by I. Kátai, B. Kovács and J. Szabó [**KK80**, **KK81**, **KS75**]. We say that a pair $(\beta, \mathcal{A})$, where β is an algebraic integer and $\mathcal{A} \subset \mathbb{Z}$ is a finite alphabet containing 0, is a *number system* (*NS*) if every $x \in \mathbb{Z}[\beta]$ has a unique representation $x = x_k x_{k-1} \cdots x_1 x_0{}_\bullet$ with $x_j \in \mathcal{A}$. We say that β satisfies the CNS property if $(\beta, \{0, 1, \dots, b-1\})$ is a number system for some $b \in \mathbb{N}$; this number system is then called a ***canonical number system*** (*CNS*). If $\beta \in \mathbb{C}$ admits $\mathcal{A}$ such that $(\beta, \mathcal{A})$ is a NS, then β is an expanding algebraic integer, i.e., all its Galois conjugates including β itself are in modulus > 1. If β has the CNS property, then neither β nor its Galois conjugates are positive real numbers. However, it is considered difficult to describe all β with the CNS property. Two classes of such β are known: First, when $P_\beta(X) = p_dX^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0$ with $1 \leqslant p_d \leqslant \cdots \leqslant p_1 \leqslant p_0$, and second, when $p_0 > p_d + \cdots + p_1$ and all $p_j \geqslant 0$ [**AR04**, **ST04**]. We note that the notion of a CNS was extended to polynomials in $\mathbb{Z}[X]$ by A. Pethő [**Pet91**], and the original definition fits in the one by Pethő if irreducible polynomials are



(Fig. 1-1) Plots of the greedy, Ito-Sadahiro and symmetric β-transformations for $|\beta| = \varphi_t = 1.839\cdots$ the Tribonacci constant.

considered only. In this context, various combinatorial and arithmetic sufficient conditions for the CNS property were given, as well as algorithms to check the property [**AP02**, **BK08**].

The similarity of the known results about the CNS property and Property (F) led to the realization that the two underlying systems can be unified, and the notion of ***shift radix systems*** (*SRS*) has been introduced by S. Akiyama et al. [**ABBPT05**]. Consider a vector $r = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$. Then the map

$$\tau_r \colon \mathbb{Z}^d \to \mathbb{Z}^d, \quad z = (z_0, \ldots, z_{d-1}) \mapsto \big(z_1, \ldots, z_{d-1}, \lfloor r \cdot z \rfloor\big),$$

where $r \cdot z$ is the dot-product $r_0 z_0 + \cdots + r_{d-1} z_{d-1}$, is called a d-*dimensional SRS*. We say that τ_r is *finite* if the zero vector $(0, \ldots, 0)$ is in the orbit of each $z \in \mathbb{Z}^d$.

Hollander used the following correspondence to prove his result on Property (F): An algebraic integer β with minimal polynomial $P_\beta(X) = X^{d+1} - a_d X^d - \cdots - a_1 X - a_0$ has Property (F) if and only if τ_r is finite, where

$$r = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \quad \text{and} \quad r_j = a_j \beta^{-1} + a_{j-1} \beta^{-2} + \cdots + a_0 \beta^{-j-1}.$$

Then, Akiyama et al. proved that β with minimal polynomial $P_\beta(X) = p_d X^d + \cdots + p_1 X + p_0$ is a CNS number if and only if τ_r is finite, where

$$r = \frac{1}{p_0}(p_d, p_{d-1}, \ldots, p_1).$$

We refer to a survey about SRS by P. Kirschenhofer and J. Thuswaldner for more details [**KT14**].

Note that when $x \in [0, 1)$, the greedy β-expansion of x can be generated as a coding by the code $D \colon [0, 1) \to \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ of the orbit of x by the transformation $T \colon [0, 1) \to [0, 1)$, $x \mapsto \beta x - D(x)$. If the code is chosen differently, and the transformation modified accordingly, we get various types of β-expansions, such as the lazy expansions which provide the lexicographically smallest expansion [**EJK90**, **DK02**, **HMP13**], the minimal weight expansions which minimize the digit sum [**FS08**], the optimal expansions which minimize the distance of the convergents to the expanded point in each step [**DdVKL12**], or the symmetric expansions whose transformation satisfies that $T(-x) = -T(x)$ for almost all x in its domain [**AS07**]. Transformations with a negative $\beta < -1$ can be considered as well [**IS09**]. Three examples of plots of β-transformations are depicted in Figure 1-1. We note that it is also possible to define transformations with complex bases that act on the complex numbers [**Pet94**, **HFI09**, **AC15**].

Property (F) can be studied also for other types of expansions than the greedy ones. We know that when a transformation has Property (F), β is either a Pisot or a Salem number, i.e., all its other Galois conjugates are $\leqslant 1$ in modulus. When β is a Pisot number, i.e., the conjugates are < 1 in modulus, Property (F) implies that the Rauzy fractals for the transformation tile the corresponding contracting hyperplane [**Aki99**, **Pra99**]. It was shown that weakening Property (F) slightly, we get a condition that is equivalent to the tiling condition [**Aki02**]. However, there exist examples when the tiles do not form a tiling of the contracting hyperplane [**KS12**]; we study this phenomenon for the symmetric expansions in Chapter 3.

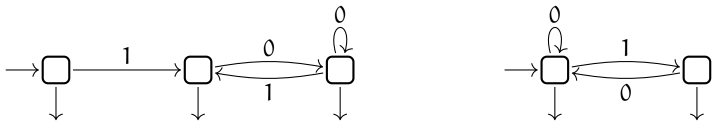
The greedy β-expansions admit an interesting arithmetical property. We know that when $\beta \in \mathbb{N}$ and $p/q \in [0, 1)$ is a rational number such that q is co-prime to β, then p/q has a purely periodic greedy β-expansion. Surprisingly, for certain quadratic β, namely roots of polynomials $\beta^2 = a\beta + 1$ with $a \geqslant 1$, we get that all rational numbers in $[0, 1)$ have a purely periodic greedy β-expansion [**Sch80**]. All Pisot units that satisfy Property (F) behave similarly [**Aki99**]. The non-unit case is more complicated. However, this arithmetical property is closely related to the shape of the Rauzy fractals [**BS07**], which allows us to study this property for general quadratic Pisot numbers, see Chapter 4.

Integer numeration systems provide another generalization of the greedy algorithm. Such a system consists of a strictly increasing sequence of integers $(b_j)_{j\geqslant 0}$, rather than of powers of β. If $b_0 = 1$, we can represent every natural number in a greedy way, with integer digits $0 \leqslant x_j < b_{j+1}/b_j$. Putting $b_j \coloneqq \beta^j$ for $\beta \in \mathbb{N}$, $\beta \geqslant 2$, we recover the standard representations.

Letting $(b_j)_{j\geqslant 0}$ be the Fibonacci sequence given by $b_1 = 1$, $b_2 = 2$, and $b_j = b_{j-1} + b_{j-2}$ for $j \geqslant 2$, we get the so-called Zeckendorf numeration [**Zec72**], in which, each number $x \in \mathbb{N}$ has a representation with digits in $\{0, 1\}$ such that two consecutive digits are never both 1's.

Various sequences $(b_j)_{j\geqslant 0}$ can be considered. A standard approach is to consider recurrence relations whose characteristic polynomial is the minimal polynomial of a Pisot number β; also, sequences related to continued fraction expansions were considered [**Ost22**].

A different approach was used by P. Lecomte and M. Rigo in the so-called ***abstract numeration systems*** [**LR01**]. They consider any infinite language over a finite ordered alphabet. Such language is totally ordered by the radix order (where shorter words are smaller than longer words and words of the same length are ordered lexicographically); The expansion of $n \in \mathbb{N}$ is then the nth



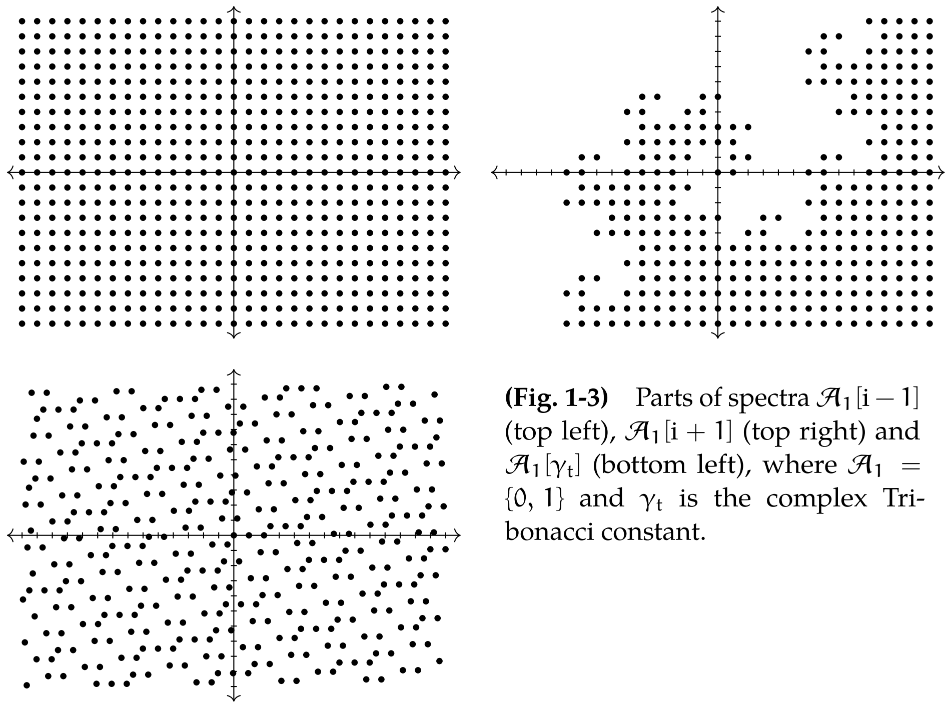
(Fig. 1-2) The acceptance automaton (graph) of the language of the Zeckendorf representations (left). Its variant that allows leading zeros (right).

smallest word in the language w.r.t. the radix order. Every integer numeration system such that b_{j+1}/b_j is bounded is an abstract numeration system; to see this, it is enough to see that the radix order on the representations preserves the order on the numbers. (But also, for instance the counting system is an abstract numeration system; here n is represented by n consecutive 1's.) The Zeckendorf system is recovered by considering all finite words with digits 0 and 1 that start with 1 and do not contain 11 as a factor; these words are labellings of paths in the graph in Figure 1-2 left that start in an in-edge and end in an out-edge. Figure 1-2 right then shows the graph for the language of the factors of these representations. Note that this is the same language as the language of the greedy β-expansions with $\beta = \varphi_g$ the golden ratio; the reason is that its minimal polynomial $X^2 - X - 1$ is the characteristic polynomial of the recurrence relation for the Fibonacci numbers. This relation has been further studied [**GT91**, **FS92**].

Another point of view on the positional numeration systems is the following: fix a base $\beta > 1$ and a finite alphabet $\mathcal{A}$, and study the properties of all representations in this system.

The consideration of only non-negative powers of β leads to the so-called ***spectra*** of numbers [**EJK90**]. A spectrum of β with alphabet $\mathcal{A}$ is the set $\mathcal{A}[\beta]$ of all polynomials with coefficients in $\mathcal{A}$, evaluated at 0. When β is a Pisot number and $\mathcal{A} \subseteq \mathbb{Z}$, the set of all possible distances between the points of spectra, i.e., $\mathcal{A}[\beta] - \mathcal{A}[\beta]$, is not dense in $\mathbb{R}$ [**Gar62**]. It is also not dense if the alphabet is too small, namely if $\#\mathcal{A} < \beta$ [**EK98**]. If $\mathcal{A} = \mathcal{A}_m := \{0, 1, \dots, m\}$ for some m, and $\beta > 1$, the spectrum can be written as an increasing sequence of numbers, $0 = x_0 < x_1 < x_2 < \cdots$. Then, the minimal distance between the points of the spectrum, $\ell_m(\beta) := \liminf_{k\to\infty}(x_{k+1} - x_k)$, is further studied; we have that $\ell_m(\beta) = 0$ if and only if $m > \beta - 1$ and β is not a Pisot number [**Bug96**, **Fen15**]. On the other hand, the spectrum $\mathcal{A}_m[\beta]$ is Delone if and only if β is a Pisot number and $m \geqslant \beta - 1$. In that case, the distances between consecutive points take only finitely many values. Also, this sequence is substitutive; roughly speaking, it can be generated by a system of rewriting rules over a finite alphabet [**FW02**]. For a particular case, $\ell_m(\beta)$ can be computed [**BH02**]. The value of $\ell_m(\beta)$ is known for all m for some particular β and classes of β [**KLP00**, **Kom02**, **BH03**].

Spectra are studied in the complex plane as well. The first example was given for the base $\gamma = i - 1$ and alphabet $\mathcal{A}_1 = \{0, 1\}$ [**Pen65**, **Knu81**], in this case, $\mathcal{A}_1[i - 1] = \mathbb{Z}[i]$, the set of Gaussian integers. Note that with the base $\gamma = i + 1$, we do not get all of $\mathbb{Z}[i]$, see Figure 1-3. Some results similar to the real case were

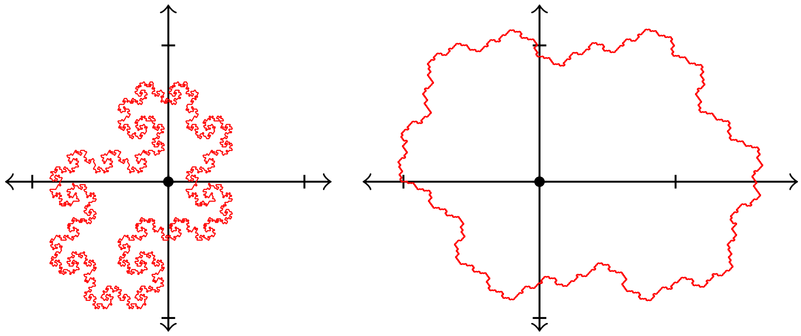


(Fig. 1-3) Parts of spectra $\mathcal{A}_1[i-1]$ (top left), $\mathcal{A}_1[i+1]$ (top right) and $\mathcal{A}_1[\gamma_t]$ (bottom left), where $\mathcal{A}_1 = \{0, 1\}$ and γ_t is the complex Tribonacci constant.

obtained, namely that $\ell_m(\gamma) = \inf_{x,y \in \mathcal{A}_m[\gamma],\, x \neq y} |x - y| > 0$ for all $m \in \mathbb{N}$ if and only if γ is a complex Pisot number. We recall that a non-real number is a *complex Pisot number* if all its Galois conjugates but itself and its complex conjugate are < 1 in modulus [**Zaï04**]. We extend this result in Chapter 5 by showing that if $m < |\gamma|^2 - 1$ then the spectrum is not relatively dense. We also provide and algorithm for computing $\ell_m(\gamma)$ for all m at once, for a class of cubic numbers γ.

The consideration of both positive and negative powers of β leads to the so-called β***-representations***. A β-representation of $x \in \mathbb{R}$ with alphabet $\mathcal{A}$ containing 0 is any string $x_{-k} \cdots x_{-1} x_0 \bullet x_1 x_2 x_3 \cdots$ with $x_j \in \mathcal{A}$ such that $x = \sum_{j \geqslant -k} x_j \beta^{-j}$. If the alphabet is too small, namely if $\#\mathcal{A} < \beta$, the set of points that have a β-representation is not an interval; for instance in the case $\beta = 3$ and $\mathcal{A} = \{0, 2\}$, it is the Cantor set. A precise condition on when representable x form an interval is known due to M. Pedicini [**Ped05**]. Besides the question of existence of a representation, we can ask how many representations exist. Restricting to representations of the form $\bullet x_1 x_2 x_3 \cdots$, N. Sidorov showed that almost every x has a continuum of expansions if and only if $\beta > \varphi_g$, the golden ratio [**Sid03a**]. He also studied the cardinality of the set of x that have less than a continuum of expansions and its Hausdorff dimension [**Sid03b**].

A lot of interest is in the study of algorithms that perform arithmetic operations such as addition. In standard addition, for instance in the decimal system, a particular digit of the result depends on all digits before it; compare, e.g., $9999998 + 2 = 10000000$ with $9999998 + 1 = 9999999$. It is possible to perform addition in the decimal system in such a way that a digit of the output depends only on a bounded part of the input; the price is that the alphabet has to enlarged



(Fig. 1-4) The twin dragon [**Knu81**] (left) and the original Rauzy fractal [**Rau82**] (right). Either of them can tile the complex plane periodically. The Rauzy fractal can also tile the plane aperiodically if we take itself and its scaled copies.

to allow redundancy [**Avi61**, **CR78**, **FPS13**]. In general, parallel addition is possible only if the base β is an algebraic number, none of whose Galois conjugates lies on the unit circle. We note that Möbius number systems provide a way of unifying the positional numeration systems with other numeration systems based on linear fractional transformations, such as the continued fraction algorithms [**Kůr09**] and that arithmetic algorithms have been also studied in the context of Möbius number systems [**Kůr12**].

Figure 1-4 shows two examples of fractals associated to positional numeration systems. The one on the left was presented by D. Knuth in relation to the complex numeration in base $i - 1$ [**Knu81**]. The one on the right was presented by G. Rauzy in relation to the Tribonacci substitution. M. Barge proved that for any greedy β-transformation with a Pisot unit β, the Thurston's tiling can be constructed [**Thu89**, **Bar15**]. Under certain conditions, general β-expansions give rise to tilings or multiple tilings as well [**KS12**]; we study the multiple tilings for the symmetric β-expansion in Chapter 3. Tilings can be also constructed for canonical number systems with monic polynomials [**KK92**] and tilings for shift radix systems have been defined as well that unify the two approaches [**BSSST11**]. Also, tilings are considered for substitutions and it is conjectured that every irreducible unimodular Pisot substitution gives a tiling by the Rauzy's construction; this open problem is known as the *Pisot conjecture* [**ABBLS15**].

Structure of the thesis

The thesis is structured in the following way. Chapter 2 recalls used notions from number theory, the theory of languages and shifts, β-expansions, model sets, and tilings. Chapters 3–5 form the core part of the thesis; in Chapter 3, results on multiple tilings for the symmetric β-transformation are presented, in Chapter 4 we discuss purely periodic Rényi expansions in quadratic bases, finally Chapter 5 comprises results on spectra of complex numbers. Each of the three main chapters is closed by its own conclusion, mentioning open problems related to its topic.

Note on authorship

The results in this thesis come mainly from three articles:

[1] Tomáš Hejda, *Multiple tilings associated to* d*-Bonacci beta-expansions*, 2015, submitted, 11 pp., `arXiv:1503.07744`.

Presented at Numeration, Nancy (FR), 2015.

Chapter 3 comprises the results of this article, together with some more results that are yet unpublished, most importantly, the general Theorem 3-1.

[2] Tomáš Hejda and Wolfgang Steiner, *Beta-expansions of rational numbers in quadratic Pisot bases*, 2014, submitted, 12 pp., `arXiv:1411.2419`.

Presented at 15^{e} Journées Montoises d'Informatique Théorique, Nancy (FR), 2014.

The article is a common work of the student and his supervisor. Chapter 4 comprises the results of this article.

[3] Tomáš Hejda and Edita Pelantová, *Spectral properties of cubic complex Pisot units*, Math. Comp. **85** (2016), no. 1, 401–421 .

Presented at Numeration and Substitution, Debrecen (HU), 2014.

The article is a common work of the student and his supervisor. Chapter 5 comprises the results of this article.

CHAPTER TWO

Preliminaries

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2-1 Number theory

The crucial notions for all results in this thesis are the notions of Pisot and complex Pisot number.

Definition 2-1. We say that an algebraic integer $\beta \in \mathbb{R}$, $\beta > 1$ is a *Pisot number* if all its Galois conjugates but β itself are < 1 in modulus.

We say that an algebraic integer $\beta \in \mathbb{C} \setminus \mathbb{R}$, $|\beta| > 1$ is a *complex Pisot number* if all its Galois conjugates but β itself and its complex conjugate $\beta^\dagger$ are < 1 in modulus.

To fix the notation, we recall that if $\beta \in \mathbb{C}$ is an algebraic number whose minimal polynomial P_β has degree d, then all the roots of this polynomial are called the *Galois conjugates* of β. One of them is $\beta_{(0)} := \beta$ itself and one of them is its complex conjugate $\beta^\dagger$ in case $\beta \notin \mathbb{R}$. Then, there is a certain number d_R of real conjugates and d_C pairs of complex conjugates; we have that $d_R + 2d_C = d - 1$ if $\beta \in \mathbb{R}$ and $d_R + 2d_C = d - 2$ if $\beta \in \mathbb{C} \setminus \mathbb{R}$. We label the real ones $\beta_{(1)}, \dots, \beta_{(d_R)}$ and the complex ones $\beta_{(d_R+1)}, \beta^\dagger_{(d_R+1)}, \dots, \beta_{(d_R+d_C)}, \beta^\dagger_{(d_R+d_C)}$.

To each $\beta_{(j)}$, a *Galois isomorphism* $\psi_{(j)} \colon \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_{(j)})$ is assigned that maps $\beta \mapsto \beta_{(j)}$. In particular, $\psi_{(0)} \colon \mathbb{Q}(\beta) \to \mathbb{Q}(\beta)$ is the identity isomorphism.

§ 2-1-1 Places of algebraic fields. Let K be an algebraic number field, and denote $\mathcal{O}_K$ its ring of integers. We consider all metrics on K, and we say that two metrics are equivalent, if they induce the same topology on K. A *place* of K is an equivalence class of metrics on K. Before the work of K. Hensel at the end of 19th century it was believed that the only places of K are the ones represented by the standard absolute value $x \mapsto |\psi_{(j)}(x)|$ for one of the Galois isomorphisms of K; these places are usually called *infinite*.

But Hensel [**Hen97**] showed that for each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, there exists an ultrametric on K associated to it, and these ultrametrics are mutually non-equivalent. Therefore there exists a place for each of the ideals, these places are usually called *finite*. The representative ultrametric for $\mathfrak{p}$ is usually defined for $x \in K$ by $|x|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-\nu_{\mathfrak{p}}(x)}$, where $\mathfrak{N}(\mathfrak{p}) \geqslant 2$ is the norm of the ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, and $\nu_{\mathfrak{p}}(x)$ is the power of $\mathfrak{p}$ in the decomposition of the principal ideal $x\mathcal{O}_K$ into prime ideals. For the precise definition of $\mathfrak{N}(\mathfrak{p})$ we refer to J. Neukirch's book [**Neu99**, Ch. III, § 1]; we will only be interested in the topological properties of K w.r.t. $|\cdot|_{\mathfrak{p}}$, the value of $\mathfrak{N}(\mathfrak{p})$ does not influence the topology.

Example 2-2. In the field $\mathbb{Q}$, the prime ideals are $\mathfrak{p} = p\mathbb{Z}$ for each rational prime $p > 0$. The associated ultrametric is then $|x|_{\mathfrak{p}} = p^{-\nu_p(x)}$, where $\nu_p(x)$ is the power of p in the prime factorization of $x \in \mathbb{Q}$.

Example 2-3. In the field $\mathbb{Q}(\sqrt{5})$, we have $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\varphi_g]$, where $\varphi_g = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$ is a principal ideal domain, therefore the prime ideals are $\mathfrak{p} = p\mathbb{Z}[\varphi_g]$ for each prime number $p \in \mathbb{Z}[\varphi_g]$ up to multiplication by units. In this field, for instance the number $p = 1 + 3\varphi_g$ is a Pisot number and a prime; for the ideal $\mathfrak{p} = p\mathbb{Z}[\varphi_g]$, we have that $|x|_{\mathfrak{p}} = 5^{-\nu_{\mathfrak{p}}(x)}$, where $\nu_p(x)$ is the power of p in the factorization of $x \in \mathbb{Q}(\sqrt{5})$. For example, $|10|_{\mathfrak{p}} = 1/25$ because of the prime factorization $10 = (5 - 3\varphi_g) \cdot 2 \cdot (1 + 3\varphi_g)^2$ (note that the first factor is a unit).

§ 2-1-2 Representation spaces. For the purpose of this section, let $\beta > 1$ be a Pisot number of degree d. Let $K = \mathbb{Q}(\beta)$, then K has $1 + d_R + d_C$ infinite places $\mathfrak{p}_{(0)}, \dots, \mathfrak{p}_{(d_R+d_C)}$. In the first one, we consider the norm $|x|_{\mathfrak{p}_{(0)}} = |x|$. In the next (real) d_R ones, we consider the norms $|x|_{\mathfrak{p}_{(j)}} = |\psi_{(j)}(x)|$. In the last (complex) d_C ones, we consider the norm $|x|_{\mathfrak{p}_{(j)}} = |\psi_{(j)}(x)|^2$.

In the view of Galois isomorphisms, we define a map

$$\Psi\colon \mathbb{Q}(\beta) \to \prod_{j=1}^{d_R+d_C} \mathbb{Q}(\beta_{(j)}), \quad x \mapsto \big(\psi_{(1)}(z), \dots, \psi_{(d_R+d_C)}(x)\big).$$

The domain of Ψ is equipped with a component-wise addition and multiplication. We can embed the domain in $\mathbb{R}^{d_R} \times \mathbb{C}^{d_C}$, which is, from the metric point of view, the same as $\mathbb{R}^{d_R+2d_C} = \mathbb{R}^{d-1}$. The convergence in K simultaneously w.r.t. the places $\mathfrak{p}_{(1)}, \dots, \mathfrak{p}_{(d_R+d_C)}$ is then the same as the convergence of the images by Ψ w.r.t. the usual topology on $\mathbb{R}^{d-1}$. We also define

$$\Psi_0\colon \mathbb{Q}(\beta) \to \mathbb{R}^d, \quad x \mapsto \big(x, \Psi(x)\big).$$

If β is not a unit, then we consider finite places of K as well, namely all prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$ for which $\beta\mathcal{O}_K \subseteq \mathfrak{p}$, which we denote $\mathfrak{p} \mid (\beta)$. We recall that an *ideal* $\mathfrak{i}$ is a non-empty subset of $\mathcal{O}_K$ such that $\mathfrak{i}\mathcal{O}_K \subseteq \mathcal{O}_K$. An ideal is a *prime ideal* if and only if its only sub-ideals are $\mathfrak{i}$ itself and $\{0\}$.

We put $K_f = \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$, where $K_\mathfrak{p}$ is the completion of K w.r.t. the norm $|\cdot|_\mathfrak{p}$. For $x \in \mathbb{Q}(\beta)$, we denote x_f the diagonal embedding of x in K_f. Last but not least, we define

$$\begin{aligned} \Psi_f \colon \mathbb{Q}(\beta) &\to \mathbb{R}^{d-1} \times K_f, \quad x \mapsto (\Psi(x), x_f), \\ \Psi_{0,f} \colon \mathbb{Q}(\beta) &\to \mathbb{R}^{d} \times K_f, \quad x \mapsto (x, \Psi(x), x_f). \end{aligned}$$

When β is a unit, we have, of course, no $\mathfrak{p}$ for which $\mathfrak{p} \mid (\beta)$, because $\mathfrak{p} \subsetneq \mathcal{O}_K = \beta\mathcal{O}_K$. We can then put $\Psi_f := \Psi$ and $\Psi_{0,f} := \Psi_0$, or we can view it as if K_f was a single point.

Example 2-4. Consider $\beta \approx 4.967$, root of $X^3 - 4X^2 - 4X - 4$. This polynomial has two other roots $\beta' \approx -0.484 + 0.756i$ and $(\beta')^\dagger$. We have that $K := \mathbb{Q}(\beta) = \{\, a + b\beta + c\beta^2 : a, b, c \in \mathbb{Q}\}$. The ideal $\beta\mathcal{O}_K$ is factored into $\beta\mathcal{O}_K = \mathfrak{i}_2\mathfrak{i}_3$, where $\mathfrak{i}_2 = 2\mathcal{O}_K + (\beta - 2)\mathcal{O}_K$ and $\mathfrak{i}_3 = 3\mathcal{O}_K + (\beta - 3)\mathcal{O}_K$ are prime ideals.

2-2 Languages and shifts

In general, we consider languages over finite alphabets. For a finite set $\mathcal{A}$ — often called *alphabet* — $\mathcal{A}^*$ denotes the set of all words over this alphabet, i.e., finite sequences of elements of the alphabet, including the empty word ε. The *length* of $u \in \mathcal{A}^*$ is denoted $|u|$. A *language* over $\mathcal{A}$ is then any subset of $\mathcal{A}^*$. The set $\mathcal{A}^*$ is naturally equipped with the operation of concatenation of words, and it forms a monoid called the *free monoid over $\mathcal{A}$*, with ε playing the role of the neutral element.

A *factor* of a finite word $u = u_0u_1 \cdots u_{n-1} \in \mathcal{A}^*$ is any finite word $w \in \mathcal{A}^*$ such that $w = u_ku_{k+1} \cdots u_{l-1}$ for some $0 \leqslant k \leqslant l \leqslant n$. If $k = 0$, we say that w is a *prefix*, if $l = n$, we say that w is a *suffix* of u. We use the notation $\operatorname{Pref}_k u$ to denote the prefix of u of length k.

A language is *regular* if it is recognized by a finite automaton; two examples of automata are in Figure 1-2 on page 5; the right one recognizes language of words over $\{0, 1\}$ that do not contain 11 as a factor; the left one adds the condition that the word does not start with 0.

To formalize the notion of automaton, we say that a *finite automaton over an alphabet $\mathcal{A}$* is a finite directed graph where parallel edges are allowed and where each edge is labelled by a letter from the alphabet. Nodes of the graph are called *states*; some states are marked as initial and some are marked as final (states can be marked both ways). A language recognized by the automaton is the set of the concatenations of the edge labellings on all path that start in an initial state and end in a final state. Where an automaton is drawn, the initial states are marked by an in-edge coming from nowhere, the final states by an out-edge to nowhere.

A special type of a finite automaton is a *finite letter-to-letter transducer from $\mathcal{A}$ to $\mathcal{B}$*. This is a finite automaton over the alphabet $\mathcal{A} \times \mathcal{B}$. It recognizes pairs of words, i.e., a relation between $\mathcal{A}^*$ and $\mathcal{B}^*$. Usually, all states in transducers are final and the out-edges to nowhere are not shown. For convenience, the first component (in $\mathcal{A}$) is called the *input* and the second one (in $\mathcal{B}$) is called the *output*.

Besides finite words, there are *infinite words* as well. We denote $\mathcal{A}^\omega$ the set of infinite words $\mathbf{u} = u_0u_1u_2\cdots$ with each $u_j \in \mathcal{A}$. For an infinite word, we define *factors* and *prefixes* accordingly.

The set $\mathcal{A}^\omega$ — when the alphabet is finite and has at least two elements — is often called a *Cantor set* or a *full shift*. A *cylinder* in $\mathcal{A}^\omega$ is any subset of $\mathcal{A}^\omega$ of the form $w\mathcal{A}^\omega$ with $w \in \mathcal{A}^*$; the cylinders $\{\, w\mathcal{A}^\omega : w \in \mathcal{A}^* \,\}$ form a base of the *Cantor topology*. The set $\mathcal{A}^\omega$ is equipped with the *Haar measure* $\mu_\mathcal{A}$, which is a probability measure defined by $\mu_\mathcal{A}(w\mathcal{A}^\omega) = (\#\mathcal{A})^{-|w|}$ for each $w \in \mathcal{A}^*$.

The *shift map* acts on $\mathcal{A}^\omega$ by $u_0u_1u_2\cdots \mapsto u_1u_2\cdots$. A *subshift* $\Sigma \subseteq \mathcal{A}^\omega$ is a subset of $\mathcal{A}^\omega$ which is closed (w.r.t. the Cantor topology) and shift-invariant. Each subshift is fully determined by a *set of forbidden factors* $F \subseteq \mathcal{A}^*$. This means that for each subshift $\Sigma \subseteq \mathcal{A}^\omega$ there exists $F \subseteq \mathcal{A}^*$ such that $\Sigma = \mathcal{A}^\omega \setminus \mathcal{A}^*F\mathcal{A}^\omega$. We say that Σ is a *sofic subshift* if there exists a regular language F that is a set of forbidden factors of Σ; we say that it is a *subshift of finite type (SFT)* if there exists a finite F. Since every finite $F \subseteq \mathcal{A}^*$ is regular, every SFT is sofic. However, the converse is not true; for instance $F = 10(00)^*1 \subseteq \{0,1\}^*$ is regular and defines a sofic subshift which contains all words in $\{0,1\}^\omega$ in which there is an even number of 0's between consecutive 1's; this subshift is not an SFT.

As the complement of a regular language is again regular, we know that the language of a sofic shift, i.e., the set $L(\Sigma) = \{\, w : w \text{ is a factor of some } u \in \Sigma \,\}$ is a regular language. Moreover, $L(\Sigma)$ is closed on taking factors and is right-extensible, i.e., for all $w \in L(\Sigma)$ we have that $wa \in L(\Sigma)$ for some $a \in \mathcal{A}$. Such a language is recognizable by a finite automaton where all states are final. The shift Σ is then the set of the labellings of infinite paths in this automaton that start in an initial state.

Note that finite letter-to-letter transducers, which recognize relations between $\mathcal{A}^*$ and $\mathcal{B}^*$, also recognize relations between $\mathcal{A}^\omega$ and $\mathcal{B}^\omega$ by considering infinite paths that start in an initial state and that visit final states infinitely many times.

2-3 Beta-expansions

Let $\beta \in \mathbb{R}$ be > 1 in modulus. Suppose $\mathcal{I} \subset \mathbb{R}$ is bounded, $\mathcal{A} \subset \mathbb{R}$ is a finite alphabet containing 0, and $D\colon \mathcal{I} \mapsto \mathcal{A}$ is a *digit function*, that is, any function satisfying that $\beta x - D(x) \in \mathcal{I}$ for all $x \in \mathcal{I}$. Then the function $T\colon \mathcal{I} \to \mathcal{I}, x \mapsto \beta x - D(x)$ is a β-*transformation*. The β-*expansion* (or T-expansion) of $x \in \mathcal{I}$ is then $x = \bullet x_1x_2x_3\cdots \in \mathcal{A}^\omega$, where $x_j = D(T^{j-1}x)$. If for $x \in \mathbb{R} \setminus \mathcal{I}$ there exists $k \in \mathbb{N}$ such that $\beta^{-k}x \in \mathcal{I}$, then we define the β-expansion of x as $x = x_{-k+1}\cdots x_0 \bullet x_1x_2\cdots$, where $\beta^{-k}x$ has the expansion $\beta^{-k}x = \bullet x_{-k+1}x_{-k+2}x_{-k+3}\cdots$, i.e., $x_j = D(T^{j+k-1}(\beta^{-k}x))$; we assume k minimal to ensure that the expansion is defined properly. Note that we index the positions in the β-expansions the other way than in (1-1) in the introduction.

We define three particular β-transformations:

The *greedy* β*-transformation* $T_G\colon [0,1) \to [0,1)$ was introduced by A. Rényi [**Rén57**], and is defined as follows:

$$T_G x := \beta x - \lfloor \beta x \rfloor; \quad \text{we have } D(x) = \lfloor \beta x \rfloor \in \mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}.$$

The *symmetric* β*-transformation* $T_G\colon [-\frac{1}{2}, \frac{1}{2}) \to [-\frac{1}{2}, \frac{1}{2})$, was introduced by S. Akiyama and K. Scheicher [**AS07**], and is defined as follows:

$$T_S x := \beta x - \lfloor \beta x + \tfrac{1}{2} \rfloor; \quad \text{we have } D(x) = \lfloor \beta x + \tfrac{1}{2} \rfloor \in \mathcal{A} = \{\lfloor \tfrac{1-\beta}{2} \rfloor, \ldots, \lceil \tfrac{\beta-1}{2} \rceil\}.$$

For $\beta \in (1,2)$, we define the *balanced* β*-transformation* $T_B\colon [\frac{2-\beta}{2\beta-2}) \to [\frac{\beta}{2\beta-2})$ as follows:

$$T_B x := \beta x - \lfloor \beta x - \tfrac{1-\beta}{2} \rfloor; \quad \text{we have } D(x) = \lfloor \beta x - \tfrac{1-\beta}{2} \rfloor \in \mathcal{A} = \{0, 1\}.$$

§ 2-3-1 Beta-transformation as a dynamical system. A *measure-preserving dynamical system* $(T, \mathcal{I}, \mu, \mathcal{B})$ consists of a domain $\mathcal{I}$, a map $T\colon \mathcal{I} \to \mathcal{I}$, a σ-algebra over $\mathcal{I}$ and a finite measure μ on $\mathcal{I}$ such that T is μ-preserving, i.e., $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$, and $\mu(\mathcal{I}) < +\infty$. We usually do not provide $\mathcal{B}$ explicitly and we assume that $\mathcal{B}$ is the σ-algebra of Borel sets on $\mathcal{I}$.

We fix an arbitrary Pisot number β of degree at least 2 and we consider the greedy and symmetric β-transformations, and in case $\beta \in (1,2)$, also the balanced one. All these transformations have a unique invariant measure that is absolutely continuous w.r.t. the Lebesgue measure (so-called *ACIM*). This follows from [**LY78**, Theorem 1]; they show that if the limits from the left are equal at all discontinuity points and also the limits from the right are equal, then there is a unique ACIM, whose support is a finite union of intervals. This is trivially satisfied by the balanced transformation that has only 1 discontinuity point, and it is satisfied by the greedy and symmetric expansions because the limits from the left and from the right are the right and left end points of the domain of the transformation, respectively. In the sequel, we consider T to be one of the three transformations T_G, T_S or T_B, and we denote μ the unique ACIM. Since our transformations are right-continuous, we consider the support of μ to consist of intervals of the form $[l, r)$.

§ 2-3-2 Rauzy fractals. To each $x \in \mathbb{Z}[\beta^{-1}] \cap \mathcal{I}$, we assign a *Rauzy fractal*, which is the Hausdorff limit

$$\mathcal{R}(x) := \operatorname*{H-lim}_{k\to\infty} \Psi\big(\beta^k T^{-k}(x)\big).$$

We have to justify that the limit exists. To this end, denote $C_n := \beta^n T^{-n}(x)$, so that $\mathcal{R}(x) = \text{H-lim}\, \Psi(C_n)$. Then

$$C_{n+1} = \beta^{n+1} T^{-1}(\beta^{-n} C_n) \subseteq \beta^n(\beta^{-n} C_n + \mathcal{A}) = C_n + \beta^n \mathcal{A},$$

because $T^{-1}S \subseteq \frac{S+\mathcal{A}}{\beta}$ for any $S \subseteq \mathcal{I}$. Also, because T is surjective, $T^{-1}(y)$ contains at least one point $x \in \mathcal{I}$ such that $y = \beta x - a$ for some $a \in \mathcal{A}$, whence $y \in \beta T^{-1}(y) - \mathcal{A}$ and $S \subseteq \beta T^{-1}(S) - \mathcal{A}$. Therefore

$$C_n = \beta^n T^{-n}(x) \subseteq \beta^n\big(\beta T^{-(n+1)}(x) - \mathcal{A}\big) = C_{n+1} - \beta^n\mathcal{A}.$$

The two relations together imply that we have, for the Hausdorff distance, $\delta(\Psi(C_{n+1}), \Psi(C_n)) \leqslant \delta(\Psi(\beta^n\mathcal{A}), \Psi(0))$. Since $\Psi(\beta^n)$ decays exponentially, the sequence $\Psi(C_n)$ is a Cauchy sequence.

§ 2-3-3 Natural extension. We now describe a model of the natural extension of the β-transformation, as given for instance by S. Ito and H. Rao [**IR05**, **IR06**]. We put

$$\mathcal{Q}_f(x) := \operatorname*{H-lim}_{n\to\infty} \Psi_f\big(x - \beta^n T^{-n}(x)\big) \subseteq \mathbb{R}^{d-1} \times K_f \quad \text{for } x \in \mathcal{I}.$$

This limit exists by the same argument as for $\mathcal{R}(x)$. An alternative definition of $\mathcal{Q}_f(x)$ is the following:

$$\mathcal{Q}_f(x) = \big\{-\textstyle\sum_{j\leqslant 0} x_j \Psi_f(\beta^{-j}) : x_{-k+1}x_{-k+2}\dots x_{-1}x_0x_1x_2\dots \text{ is T-admissible} \\ \text{for all } k \geqslant 0\big\},$$

where $\bullet x_1x_2x_3\cdots$ is the T-expansion of x; the sum in the above formula is convergent, because $\Psi_f(\beta^j) \to \Psi_f(0)$ as $j \to \infty$. Since the language of T-admissible expansions is rational, we immediately get that $\mathcal{Q}_f(x)$ takes only finitely many shapes.

But not only that, we also have that the domain $\mathcal{I}$ is split into disjoint subintervals $\mathcal{I} = \bigcup_{v\in V}[v, \hat{v})$, where $V \subset \mathcal{I}$ is a finite index set, such that $x \mapsto \mathcal{Q}_f(x)$ is constant on each $[v, \hat{v})$. This has been first pointed out by W. Thurston [**Thu89**], proof of the statement is given for instance by C. Kalle and W. Steiner [**KS12**, Proposition 3.9] in the case β unit, and by M. Minervino and W. Steiner [**MS14**, § 2.2 and Theorem 2] in the general case. Following these two papers, we define the natural extension $(\mathcal{T}, \mathcal{X}, \lambda^d \times \mu_f)$ of $(T, \mathcal{I}, \mu)$ as follows:

$$\mathcal{X} := \bigcup_{v\in V} [v, \hat{v}) \times \mathcal{Q}_f(v) \subseteq \mathbb{R}^d \times K_f,$$
$$\mathcal{T} : \mathcal{X} \to \mathcal{X}, \quad (x, y) \mapsto \big(Tx, \Psi(\beta)y - \Psi(D(x))\big),$$

where we recall that $D(x) = \beta x - Tx$ is the first digit of the expansion of x, λ^d is the Lebesgue measure on $\mathbb{R}^d$, and μ_f is the product measure of the Haar measures on each $K_\mathfrak{p}$ for $\mathfrak{p} \mid (\beta)$. (Note that we can define μ_f in terms of the Haar measure on $\mathcal{A}^\omega$ with a suitable alphabet $\mathcal{A}$, see § 4-4).

We also define

$$\mathcal{Q}(x) := \operatorname*{H-lim}_{n\to\infty} \Psi\big(x - \beta^n T^{-n}(x)\big) \subseteq \mathbb{R}^{d-1} \quad \text{for } x \in \mathcal{I}$$

and we have that $\mathcal{Q}(x) = x - \mathcal{R}(x)$ for all $x \in \mathbb{Z}[\beta^{-1}] \cap \mathcal{I}$. We recall that for β unit, we have that K_f is a single point, and $\mathcal{Q}_f(x)$ coincides with $\mathcal{Q}(x)$.

Denoting π_0 the projection $\mathbb{R}^d \times K_f \to \mathbb{R}$, $(x, y) \mapsto x$ (for $x \in \mathbb{R}$ and $y \in \mathbb{R}^{d-1} \times K_f$), we get that $\pi_0(\mathcal{T}(x, y)) = T(\pi_0(x, y))$ for all $(x, y) \in \mathcal{T}$, whence $d\mu(x)/dx = \int_{y \in \mathcal{Q}_f(x)} d(\lambda^{d-1} \times \mu_f)$. This means that if x does not lie in the support of the invariant measure, then $\mathcal{Q}_f(x)$ is a set of zero measure.

2-4 Model sets

Model sets appear naturally in the context of Rauzy fractals for β-transformation, where β is a degree d Pisot number, since the set $\Psi(\mathbb{Z}[\beta] \cap \mathcal{I})$ is a model set with physical space $\mathbb{R}^{d-1}$ and internal space $\mathbb{R}$.

Definition 2-5. A *model (cut-and-project) set* is the set

$$\Lambda_{\Phi_I, \Phi_P}(\Omega) := \{\, \Phi_P(z) : \Phi_I(z) \in \Omega,\ z \in \mathbb{Z}^d \,\},$$

where:

- $d, d_P, d_I \geqslant 1$ are integers such that $d = d_P + d_I$;
- the linear map $\Phi_P \colon \mathbb{R}^d \to \mathbb{R}^{d_P}$ is onto and its restriction to $\mathbb{Z}^d$ is injective;
- the linear map $\Phi_I \colon \mathbb{R}^d \to \mathbb{R}^{d_I}$ is such that $\Phi_I(\mathbb{Z}^d)$ is dense in $\mathbb{R}^{d_I}$;
- the linear map $\mathbb{R}^d \to \mathbb{R}^d$, $z \mapsto (\Phi_P(z), \Phi_I(z))$ has the full rank d;
- the set $\Omega \subset \mathbb{R}^{d_I}$ is bounded and the closure of Ω is equal to the closure of its interior.

The space $\mathbb{R}^{d_P}$ is called the *physical space*, the space $\mathbb{R}^{d_I}$ is called the *internal space* and the set Ω is called the *(acceptance) window*. Since $\Phi_P|_{\mathbb{Z}^d}$ is injective, we can define a map $\star \colon \Phi_P(\mathbb{Z}^d) \to \mathbb{R}^{d_I}$ as $x^\star := \Phi_I(\Phi_P^{-1}(x))$.

For the case when β is a degree d Pisot number, we put

$$\begin{aligned} \Phi_I \colon \mathbb{R}^d \to \mathbb{R}, \qquad & (z_0, \ldots, z_{d-1}) \mapsto z_0 + z_1\beta + \cdots + z_{d-1}\beta^{d-1}, \\ \Phi_P \colon \mathbb{R}^d \to \mathbb{R}^{d-1}, \quad & (z_0, \ldots, z_{d-1}) \mapsto z_0\Psi(1) + z_1\Psi(\beta) + \cdots + z_{d-1}\Psi(\beta^{d-1}). \end{aligned}$$

If we restrict Φ_P to $\mathbb{Z}^d$, we get that $\Phi_P|_{\mathbb{Z}^d} = \Psi \circ \Phi_I|_{\mathbb{Z}^d}$, whence $x^\star = \Psi^{-1}(x)$ for all $x \in \Phi_I(\mathbb{Z}^d)$. We also have that $\Phi_I(\mathbb{Z}^d) = \mathbb{Z}[\beta]$. Therefore we get that

$$\Lambda_\beta(\Omega) := \{\, \Psi(z) : z \in \mathbb{Z}[\beta] \cap \Omega \,\} \tag{2-1}$$

is a model set. The relation between the sets and the mappings can be seen in the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{R}^d \arrow[d, phantom, "\cup"] \arrow[ddl, "\Phi_I"'] \arrow[ddr, "\Phi_P"] & \\
 & \mathbb{Z}^d \arrow[dl, "\Phi_I"'] \arrow[dr, "\Phi_P"] & \\
\mathbb{R} \supset \mathbb{Z}(\beta) \arrow[rr, shift left, "\Psi"] & & \Psi(\mathbb{Z}(\beta)) \subset \mathbb{R}^{d-1} \arrow[ll, shift left, "\star"]
\end{tikzcd}$$

It is well known that model sets are special cases of Meyer sets, which themselves are Delone sets. We say that a set $\Lambda \subset F := \mathbb{R}^d$ is:

- *uniformly discrete* if there exists $r > 0$ such that $|x - y| \geqslant r$ for all distinct $x, y \in \Lambda$;
- *relatively dense* if there exists $R > 0$ such that for all $x \in F$, the ball $B_{R/2}(x) := \{ z \in F : |z - x| \leqslant R/2 \}$ meets Λ, i.e., contains a point $x \in \Lambda$.
- a *Delone set* if it is both uniformly discrete and relatively dense.
- a *Meyer set* if both Λ and $\Lambda - \Lambda = \{ x - y : x, y \in \Lambda \}$ are Delone.

The concept of Meyer sets was considered by Y. Meyer in 1972 [**Mey72**]. The fundamental study of model and Meyer sets is in R. Moody's paper from 1997 [**Moo97**]. Model sets have a large number of interesting properties.

Consider a model set $\Lambda = \Lambda_{\Phi_I, \Phi_P}[l, r)$. Then we have the following:

1. Λ is *repetitive*; consider any ρ-*patch* $\Pi_\rho(x) := \Lambda \cap B_\rho(x)$ for $x \in \Lambda$ and $\rho > 0$, then $\Pi_\rho(x) - x = \Pi_\rho(y) - y$ for infinitely many $y \in \Lambda$. Moreover, for any fixed x, ρ, the set of y such that $\Pi_\rho(x) - x = \Pi_\rho(y) - y$ is again a model set, just with a different acceptance window.
2. Λ has *finite local complexity*; for a fixed $\rho > 0$, the set of different ρ-patches $\{ \Pi_\rho(x) - x : x \in \Lambda \}$ is finite.

In Item 1 above, we can say even more about the new acceptance window; this statement will be useful later:

Lemma 2-6. *Let $\Lambda(\Omega) = \Lambda_{\Phi_I, \Phi_P}(\Omega)$ be a model set with $\Omega = [l, r)$. Consider any $x \in \Lambda(\mathbb{R})$ and $\rho > 0$, where ρ is large enough so that $\Pi_\rho(x) \neq \emptyset$. Then there exist $\varepsilon_1 \leqslant 0 < \varepsilon_2$ such that*

$$\{ y \in \Lambda(\mathbb{R}) : \Pi_\rho(x) - x = \Pi_\rho(y) - y \} = x + \Lambda[\varepsilon_1, \varepsilon_2). \tag{2-2}$$

Proof. We put

$$\begin{aligned} \varepsilon_1 &:= -\min\{ z^\star - l + x^\star : z \in \Phi_P(\mathbb{Z}^d) \cap B_\rho(0),\ z^\star \geqslant l - x^\star \} \\ &\qquad \cup \{ z^\star - r + x^\star : z \in \Phi_P(\mathbb{Z}^d) \cap B_\rho(0),\ z^\star \geqslant r - x^\star \}, \\ \varepsilon_2 &:= -\max\{ z^\star - l + x^\star : z \in \Phi_P(\mathbb{Z}^d) \cap B_\rho(0),\ z^\star < l - x^\star \} \\ &\qquad \cup \{ z^\star - r + x^\star : z \in \Phi_P(\mathbb{Z}^d) \cap B_\rho(0),\ z^\star < r - x^\star \}. \end{aligned} \tag{2-3}$$

Obviously $\varepsilon_1 \leqslant 0 < \varepsilon_2$. Consider $y \in \Lambda(\mathbb{R})$. Then $\Pi_\rho(y) - y = \Lambda[l - y^\star, r - y^\star) \cap B_\rho(0)$. First, suppose that y satisfies (2-2), then (2-3) guarantees that $\Lambda[l - y^\star, r - y^\star) \cap B_\rho(0) = \Lambda[l - x^\star, r - x^\star) \cap B_\rho(0)$, as desired. Second, if y does not satisfy (2-2), then $\Lambda[l - y^\star, r - y^\star) \cap B_\rho(0)$ and $\Lambda[l - x^\star, r - x^\star) \cap B_\rho(0)$ differ in at least one point. ∎

2-5 Tilings and multiple tilings

We define tilings only for Euclidean spaces $\mathbb{R}^n$. In Chapter 4, we work with Rauzy fractals in $\mathbb{R}^n \times K_f$, where K_f has the topology of the Cantor space; however, we do not rely on the tiling properties of these Rauzy fractals.

Definition 2-7. Let $\mathcal{T} = \{\mathcal{T}(x)\}_{x \in X}$ be a countable collection of sets $\mathcal{T}(x) \subset \mathbb{R}^n$. We say that $\mathcal{T}$ is a *multiple tiling of* $\mathbb{R}^n$ *of covering degree* m, for $m \geqslant 1$, if the following conditions are satisfied:

1. The sets $\mathcal{T}(x)$ take only finitely many shapes: there are only finitely many classes of $\mathcal{T}$ modulo the group of translations in $\mathbb{R}^n$.
2. The family $\mathcal{T}$ is *locally finite*: for any compact $C \subset \mathbb{R}^n$, the set of tiles that meet C, i.e., $\{x \in X : \mathcal{T}(x) \cap C \neq \emptyset\}$, is finite.
3. Every $\mathcal{T}(x)$ is compact and it is a closure of its interior.
4. Almost every $y \in \mathbb{R}^n$ is contained in exactly m tiles.

We say that $\mathcal{T}$ is a *tiling* if it is a multiple tiling of covering degree 1.

Tilings appear in several contexts. First, the Rauzy fractals as defined in § 2-3-2 form a multiple tiling for Pisot units β, whenever the β-transformation T has some reasonable properties:

Theorem 2-8 [KS12, Theorem 4.10]. *Let* $T\colon \mathcal{I} \to \mathcal{I}$ *be a* β*-transformation with a Pisot unit* β *and let* μ *be its ergodic invariant measure (as in § 2-3-1). Suppose that the alphabet* $\mathcal{A}$ *satisfies* $\mathcal{A} \subset \mathbb{Z}[\beta]$*, the support of* μ *is the whole* $\mathcal{I}$ *and the language of* T*-admissible sequences is regular. Then the collection of Rauzy fractals* $\{\mathcal{R}(x)\}_{x \in \mathbb{Z}[\beta]}$ *forms a multiple tiling of* $\mathbb{R}^{d-1}$.

It had been conjectured since the beginning of the idea of fractals associated to numeration systems in 1980s [**Rau82**, **Thu89**] that for the Rényi expansions, the Rauzy fractals form a tiling. This is known as the *Pisot conjecture for* β*-numeration* and it has been proved recently by M. Barge [**Bar16**, **Bar15**].

Besides the tilings by the Rauzy fractals, we consider the Voronoi tilings as well:

Definition 2-9. Suppose $\Lambda \in \mathbb{R}^n$ is a Delone set. The *Voronoi tile* of $x \in \Lambda$ is the set of points that are closer (or at equal distance) to x than to $\Lambda \setminus \{x\}$:

$$\mathcal{V}(x) := \{ z \in \mathbb{R}^n : |z - x| \leqslant |z - y| \text{ for all } y \in \Sigma \}.$$

The *Voronoi tiling induced by* Λ is then the collection $\{\mathcal{V}(x)\}_{x \in \Lambda}$.

An example of a multiple tiling by Rauzy fractals is given in Figure 3-3 on page 25, and example of a Voronoi tiling is given in Figure 5-3 on page 54.

CHAPTER THREE

Multiple tilings

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3-1 Introduction

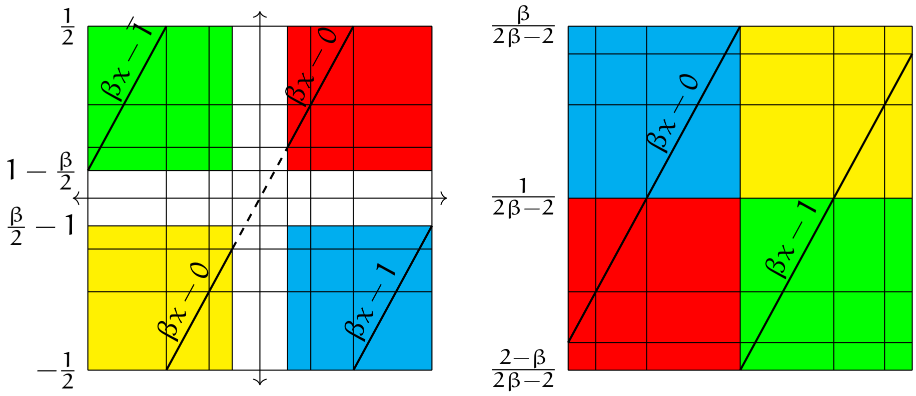
The idea of using a β-transformation and its related numeration system to generate a tiling goes back to 1980s and the works of G. Rauzy [**Rau82**] and W. Thurston [**Thu89**]. The result of Rauzy can be interpreted as that the greedy β-transformation for $\beta = \varphi_t$ the Tribonacci number induces a tiling of $\mathbb{R}^2$. S. Akiyama [**Aki99**] showed that when β has Property (F) then the transformation induces a tiling with 0 as an interior point of $\mathcal{R}(0)$. Akiyama [**Aki02**, Proposition 2] showed that the tiling property is equivalent to Property (W) (the *weak finiteness property*), i.e., that for all $x \in \mathbb{Z}[\beta^{-1}] \cap \mathcal{I}$ and all $\varepsilon > 0$ there exist y, z whose expansion is finite, such that $|y| < \varepsilon$ and $x = y - z$. There are more properties which are equivalent to the tiling property [**ABEI01**, **Sie04**, **IR06**]. M. Barge [**Bar16**, **Bar15**] has recently proved that every greedy β-transformation with a Pisot number β satisfies the tiling property.

On the other hand, C. Kalle and W. Steiner [**KS12**] gave examples of β such that the symmetric β-transformation induces a multiple tiling of covering degree 2. One example is for $\beta = \varphi_t$, the Tribonaccci constant; we generalize this result to all d-Bonacci numbers, see Theorem 3-2. Another example is for $\beta = \varphi_p$, the minimal Pisot number; we comment on this example in § 3-5.

3-2 Main results

In this chapter, two main results are proved. First the degree of the multiple tiling for the symmetric β-expansions is discussed for all Pisot units $\beta \in (1, 2)$; this is then applied to the case of d-Bonacci numbers.

First, note that the support of the ACIM μ_S of the symmetric β-transformation $T_S x = \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$ is certainly a subset of $\mathcal{I}_S \coloneqq [-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$; to see this, we note that $T_S \mathcal{I}_S = \mathcal{I}_S$ and for any non-zero $x \in [\frac{\beta}{2} - 1, 1 - \frac{\beta}{2})$ we have that $T_S^k x \in \mathcal{I}_S$ for some $k \in \mathbb{N}$. Therefore we have, for the Rauzy fractals



(Fig. 3-1) Plots of the symmetric and balanced β-transformations for β the Tribonacci constant.

for the symmetric β-transformation — denoted $\mathcal{R}_S$, that $\mathcal{R}_S(x) = \{\Psi(x)\}$ for all $x \in [-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{Z}[\beta]$ such that $x \notin \mathcal{I}_S$. We denote m_S the degree of the multiple tiling $\{\mathcal{R}_S(x)\}_{x\in\mathbb{Z}[\beta]\cap\mathcal{I}_S}$.

Note that Theorem 2-8 guarantees that $\{\mathcal{R}_S(x)\}_{x\in\mathbb{Z}[\beta]\cap\operatorname{supp}\mu_S}$ is a multiple tiling, where $\operatorname{supp}\mu_S \subset \mathcal{I}_S$ is the support of μ_S. However, for $x \in \mathbb{Z}[\beta] \cap \mathcal{I}_S$ such that $x \notin \mathcal{I}_S$, we know that $\mathcal{R}_S(x)$ has measure zero, therefore it does not contribute to the multiple tiling. We allow this little imprecision to simplify the arguments.

We also recall that the balanced β-transformation, for $\beta \in (1,2)$, is the map $T_B x = \beta x - \lfloor \beta x + \frac{2-\beta}{2\beta-2} \rfloor$ defined on $\mathcal{I}_B := [\frac{2-\beta}{2\beta-2}, \frac{\beta}{2\beta-2})$. We denote μ_B, $\mathcal{R}_B(x)$ and m_B accordingly. We prove the following statement:

Theorem 3-1. *Suppose $\beta \in (1,2)$ is a Pisot unit. Then the degrees of the multiple tilings for the symmetric and balanced expansion satisfy that*

$$m_S = \left|N(\beta-1)\right| m_B, \tag{3-1}$$

where $N(\beta-1) \in \mathbb{Z}$ denotes the norm of the algebraic integer $\beta - 1$.

In particular, $\{\mathcal{R}_S(x)\}_{x\in\mathbb{Z}[\beta]\cap\mathcal{I}_S}$ forms a tiling if and only if $\{\mathcal{R}_B(x)\}_{x\in\mathbb{Z}[\beta]\cap\mathcal{I}_B}$ forms a tiling and $\beta - 1$ is a unit.

Note that if P_β is the minimal polynomial of β, then we have that $|N(\beta-1)| = |P_\beta(1)|$. For the d-Bonacci numbers, we give the degree m_S explicitly:

Theorem 3-2. *Let $d \geqslant 2$ and let β be the d-Bonacci number, i.e., the Pisot root of $\beta^d = \beta^{d-1} + \beta^{d-2} + \cdots + \beta + 1$. Then the degree of the multiple tiling for the symmetric β-expansions is*

$$m_S = d - 1.$$

3-3 Relation between symmetric and balanced expansions

We establish a strong relation between the symmetric and the balanced expansions, in the case when $\beta \in (1,2)$. We recall that two measure-preserving dynamical systems $(\mathcal{I}_1, T_1, \mu_1)$ and $(\mathcal{I}_2, T_2, \mu_2)$ are *conjugate* iff there exists a one-to-one correspondence $\eta\colon \mathcal{I}_1 \to \mathcal{I}_2$ such that $T_1 = \eta^{-1} T_2 \eta$ and $\mu_1(B) = \mu_2(\eta B)$ for all measurable $B \subseteq \mathcal{I}_1$; the map η is called *conjugation*. Note that a conjugation in this sense is more than an isomorphism since for an isomorphism, we only require that $T_1 x = \eta^{-1} T_2 \eta x$ for μ_1-almost every $x \in \mathcal{I}_1$.

It is easy to see that the transformations $(T_S, \mathcal{I}_S)$ and $(T_B, \mathcal{I}_B)$ are conjugate:

Observation 3-3. *Denote*

$$\eta\colon \mathcal{I}_S \to \mathcal{I}_B, \quad x \mapsto \begin{cases} \frac{1}{\beta-1} x & \text{if } x \in [1 - \frac{\beta}{2}, \frac{1}{2}), \\ \frac{1}{\beta-1}(x+1) & \text{if } x \in [-\frac{1}{2}, \frac{\beta}{2} - 1). \end{cases} \tag{3-2}$$

Then η *is a conjugation* $(\mathcal{I}_S, T_S) \to (\mathcal{I}_B, T_B)$*, i.e.,* $T_S = \eta^{-1} T_B \eta$.

Proposition 3-4. *Let* $y \in \mathbb{Z}[\beta] \cap \mathcal{I}_B$. *Then*

$$\Psi(\beta - 1)\mathcal{R}_B(y) = \mathcal{R}_S(\eta^{-1} y).$$

Proof. We have that

$$\mathcal{R}_S(\eta^{-1} y) = \operatorname*{H-lim}_{n\to\infty} \Psi(C_n), \quad \text{where } C_n := \beta^n T_S^{-n} \eta^{-1} y.$$

Defining $\iota\colon \mathcal{I}_B \to \{0, 1\}$ by $\eta^{-1} x = (\beta - 1)x - \iota(x)$, we get that

$$C_n = \beta^n \eta^{-1} T_B^{-n} y = \{\, (\beta - 1)\beta^n z - \beta^n \iota(z) : z \in T_B^{-n} y \,\}.$$

From this we see that for each point $z \in C_n$, $\Psi(z)$ or $\Psi(z) + \Psi(\beta^n)$ is in $\Psi((\beta - 1)\beta^n T_B^{-n} y)$, and vice versa: for each point $\Psi(z) \in \Psi((\beta - 1)\beta^n T_B^{-n} y)$, z or $z - \beta^n$ is in C_n. Since $\lim_{n\to\infty} \Psi(\beta^n) = \Psi(0)$, we get that

$$\mathcal{R}_S(\eta^{-1} y) = \text{H-lim}\, \Psi(C_n) = \text{H-lim}\, \Psi((\beta - 1)\beta^n T_B^{-n} y) = \Psi(\beta - 1)\mathcal{R}_B(y). \quad \square$$

Since $z \mapsto \Psi(\beta - 1)z$ is a linear bijection $\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$, we directly obtain the following result:

Corollary 3-5. *The two collections of tiles*

$$\{\, \mathcal{R}_S(x) : x \in \eta^{-1}(\mathbb{Z}[\beta] \cap \mathcal{I}_B) \,\} \quad \text{and} \quad \{\, \mathcal{R}_B(x) : x \in \mathbb{Z}[\beta] \cap \mathcal{I}_B \,\} \tag{3-3}$$

form a multiple tiling of the same covering degree m_B.

Proof of Theorem 3-1. The proof is based on the fact that for any algebraic integer α, the congruence mod α on $\mathbb{Z}[\alpha]$ has exactly $|N(\alpha)|$ classes. Let H be a set of representatives of the classes modulo $\beta - 1$ such that $0 \in H$. We will show that for each $h \in H$, the collection of tiles

$$\{\, \mathcal{R}_S(x) : x \in L_h \,\}$$

is a multiple tiling of degree m_B, where we denote, for convenience,

$$L_h := \eta^{-1}\Big(\frac{h}{\beta-1} + \mathbb{Z}[\beta]\Big). \tag{3-4}$$

With this notation, we know from Corollary 3-5 that $\{\mathcal{R}_S(x) : x \in L_0\}$ is a multiple tiling of covering degree m_B.

We recall from § 2-3-1 that $\mathcal{I}_S$ is a finite union of intervals $\mathcal{I}_S = \bigcup_{\nu\in V}[\nu,\hat\nu)$ such that for each $x \in \mathbb{Z}[\beta] \cap [\nu,\hat\nu)$ we have that $\mathcal{R}_S(x) = \Psi(x) - \mathcal{Q}_S(\nu)$. We also recall that $\Psi(\mathbb{Z}[\beta] \cap [\nu,\hat\nu))$ is a model set according to (2-1), we have

$$\Psi\big(\mathbb{Z}[\beta] \cap [\nu,\hat\nu)\big) = \Lambda_\beta[\nu,\hat\nu).$$

Let $\mathrm{Pat}_\rho(0) := \{\mathcal{R}_S(y) : y \in \mathcal{I}_S,\ \Psi(y) \in B_\rho(0)\}$ be a patch of the multiple tiling for T_S centered at 0. We have

$$\mathrm{Pat}_\rho(0) = \bigcup_{\nu\in V}\{\mathcal{R}_S(y) : \Psi(y) \in \Pi_{\nu,\rho}(0)\},$$

where

$$\Pi_{\nu,\rho}(0) := \big\{\Psi(y) : y \in \mathbb{Z}[\beta] \cap [\nu,\hat\nu),\ \Psi(y) \in B_\rho(0)\big\}$$

(note that we distinguish patches of the underlying model set — i.e., collections of points of model sets — denoted $\Pi_{\nu,\rho}(0)$; and patches of the multiple tilings — i.e., collections of tiles — denoted $\mathrm{Pat}_\rho(0)$). We can apply Lemma 2-6 here to each of the model sets $\Lambda[\nu,\hat\nu)$ and obtain $\varepsilon_{\nu,1}$ and $\varepsilon_{\nu,2}$ for each $\nu \in V$. We put $\varepsilon_2 := \min_{\nu\in V}\varepsilon_{\nu,2}$.

For each $h \in H$, choose $w_h \in -h + (\beta-1)\mathbb{Z}[\beta]$ such that $\Psi(w_h) \in \Lambda[0,\varepsilon_2)$. Then $\Pi_{\nu,\rho}(0) = \Pi_{\nu,\rho}(\Psi(w_h)) - \Psi(w_h)$ and

$$\Pi_{\nu,\rho}(0) \cap \Psi(L_h) = \Pi_{\nu,\rho}(\Psi(w_h)) \cap \Psi(L_0) - \Psi(w_h).$$

From this, we derive:

$$\begin{aligned}
\mathrm{Pat}_\rho(0) &= \bigcup_{h\in H}\bigcup_{\nu\in V}\{\Psi(x) - \mathcal{Q}_S(\nu) : x \in L_h,\ \Psi(y) \in \Pi_{\nu,\rho}(0)\}\\
&= \bigcup_{h\in H}\bigcup_{\nu\in V}\{\Psi(y) - \mathcal{Q}_S(\nu) - \Psi(w_h) : y \in L,\ \Psi(y) \in \Pi_{\nu,\rho}(w_h)\}\\
&= \bigcup_{h\in H}\{\mathcal{R} - \Psi(w_h) : \mathcal{R} \in \mathrm{Pat}_{0,\rho}(w_h)\},
\end{aligned}$$

where $\mathrm{Pat}_{0,\rho}(w) = \{\mathcal{R}_S(y) : y \in L_0, \Psi(x) \in \Pi_\rho(w)\}$ is a patch of the multiple tiling $\{\mathcal{R}_S(x) : x \in L_0\}$. This means that $\mathrm{Pat}_\rho(0)$ is a union of $\#H$ translations of patches of $\{\mathcal{R}_S(x) : x \in L_0\}$ Since $\#H = |N(\beta-1)|$ and $\{\mathcal{R}_S(x) : x \in L_0\}$ has covering degree m_B, this completes the proof. □

Remark 3-6. We can easily write out what L_h is directly, we have that

$$L_h = \big((h + (\beta-1)\mathbb{Z}[\beta]) \cap [1-\tfrac{\beta}{2},\tfrac{1}{2})\big) \cup \big((h - 1 + (\beta-1)\mathbb{Z}[\beta]) \cap [-\tfrac{1}{2},\tfrac{\beta}{2}-1)\big).$$

In case when $m_B = 1$, we obtain that the collection $\{\mathcal{R}_S(x) : x \in L_h\}$ forms a tiling of $\mathbb{R}^{d-1}$ for each $h \in \{0,1,\dots,|N(\beta-1)|-1\}$.

3-4 The d-Bonacci case. Proof of Theorem 3-2

For a d-Bonacci number, whose minimal polynomial is $P_\beta(X) = X^d - \sum_{j=0}^{d-1} X^j$, we have that

$$|N(\beta - 1)| = |P_\beta(1)| = (d-1). \tag{3-5}$$

Therefore, in order to prove Theorem 3-2, we only need to show that T_B induces a tiling:

Proposition 3-7. *Let β be a d-Bonacci number for $d \geqslant 2$. Then the collection of tiles $\{\mathcal{R}_B(x) : x \in \mathbb{Z}[\beta] \cap \mathcal{I}_B\}$ is a tiling.*

We use the following statement, where we denote $\mathcal{P}_B$ the set of $y \in \mathcal{I}_B \cap \mathbb{Z}[\beta]$ that have a purely periodic balanced expansion:

Lemma 3-8 [KS12, Proposition 4.15]. *Suppose $z \in \mathbb{Z}[\beta] \cap [0, \infty)$. Let $k \in \mathbb{N}$ be an integer such that for all $y \in \mathcal{P}_B$, the expansions of y and $y + \beta^{-k}z$ have a common prefix at least as long as the period of y.*

Then $\Phi(z)$ lies in a tile $\mathcal{R}_B(x)$ for $x \in \mathbb{Z}[\beta] \cap \mathcal{I}_B$ if and only if

$$x = T_B^k(y + \beta^{-k}z) \quad \textit{for some } y \in \mathcal{P}_B.$$

Lemma 3-9. *A sequence $\bullet x_1x_2x_3\cdots$ is a balanced d-Bonacci expansion of some $x \in \mathcal{I}_B$ if and only if it does not contain 0^{d+1} nor 1^{d+1} as a factor, and for all $j \in \mathbb{N}$, $x_{j+1}x_{j+2}\cdots \neq (1^d0)^\omega$.*

Proof. Denote $l := \frac{2-\beta}{2\beta-2}$ and $b(x) \in \{0,1\}^\omega$ the balanced expansion of $x \in [l, l+1)$. We have that

$$b(l) = (0^d1)^\omega, \quad \lim_{\varepsilon \searrow 0} b(l + \tfrac{1}{2} - \varepsilon) = (01^d)^\omega,$$
$$b(l + \tfrac{1}{2}) = (10^d)^\omega, \quad \lim_{\varepsilon \searrow 0} b(l + 1 - \varepsilon) = (1^d0)^\omega.$$

According to [KS12, Theorem 2.5], a string $x_1x_2x_3\cdots$ is the balanced expansion of some $x \in [l, l+1)$ if and only if for all $j \in \mathbb{N}$ we have

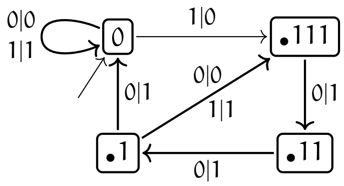
$$(0^d1)^\omega \overset{(A_j)}{\preceq} x_{j+1}x_{j+2}\cdots \overset{(B_j)}{\prec} (01^d)^\omega \quad \text{if } x_{j+1} = 0,$$
$$(10^d)^\omega \overset{(C_j)}{\preceq} x_{j+1}x_{j+2}\cdots \overset{(D_j)}{\prec} (1^d0)^\omega \quad \text{if } x_{j+1} = 0,$$

where $\prec$ and $\preceq$ denotes the lexicographic ordering on $\mathcal{A}^\omega$ with $\mathcal{A} = \{0,1\}$: We have that $u_1u_2\cdots \prec v_1v_2\cdots$ if there exists k such that $u_k \neq v_k$ and $u_k < v_k$ for the smallest such k. We have that $\mathbf{u} \preceq \mathbf{v}$ if $\mathbf{u} \prec \mathbf{v}$ or $\mathbf{u} = \mathbf{v}$.

If $x_{j+1} = 0$, then $x_{j+2} = 0 \Rightarrow (B_j)$ and $x_{j+2} = 1 \Rightarrow \big((B_j) \Leftrightarrow (D_{j+1})\big)$. Similarly, if $x_{j+1} = 1$, then $x_{j+2} = 1 \Rightarrow (C_j)$ and $x_{j+2} = 0 \Rightarrow \big((C_j) \Leftrightarrow (A_{j+1})\big)$. From this we get that only (A_j) and (D_j) have to be verified. We have that

$$(A_j) \iff x_{j+1}x_{j+2}\cdots \notin (0^d1)^*0^{d+1}\mathcal{A}^\omega,$$
$$(D_j) \iff x_{j+1}x_{j+2}\cdots \notin (1^d0)^*1^{d+1}\mathcal{A}^\omega \cup \{(1^d0)^\omega\}.$$

This is equivalent to the statement of the lemma. □



(Fig. 3-2) Transducer accepting the greedy expansion of $x \in [l, 1)$ on the input and the balanced one of it on the output; the Tribonacci case.

Lemma 3-10. *For the balanced* d*-Bonacci expansions, we have that*

$$\mathcal{P}_B = \{ \bullet(1^{n-1}01^{d-n})^\omega : 1 \leqslant n \leqslant d \} = \{ \bullet 1^n : 1 \leqslant n \leqslant d \}.$$

Proof. Consider any $x \in \mathbb{Z}[\beta] \cap [l, l+1)$, where we denote $l := \frac{2-\beta}{2\beta-2}$ so that $\mathcal{I}_B = [l, l+1)$. Since d-Bonacci numbers have Property (F) for greedy expansions [**FS92**], we know that each $x \in \mathbb{Z}[\beta] \cap \mathcal{I}_B \cap [0, 1) = \mathbb{Z}[\beta] \cap [l, 1)$ has a finite representation $x = \bullet x_1 x_2 \ldots x_k$ with $x_k = 1$, and we put $x_i := 0$ for all $i \geqslant k$, for convenience. Suppose $x = \bullet y_1 y_2 y_3 \cdots$ is the balanced expansion of x. Denote

$$s_i := \underbrace{\bullet y_{i+1} y_{i+2} y_{i+3} \cdots}_{\in [l, l+1)} - \underbrace{\bullet x_{i+1} x_{i+2} x_{i+3} \cdots}_{\in [0,1)}.$$

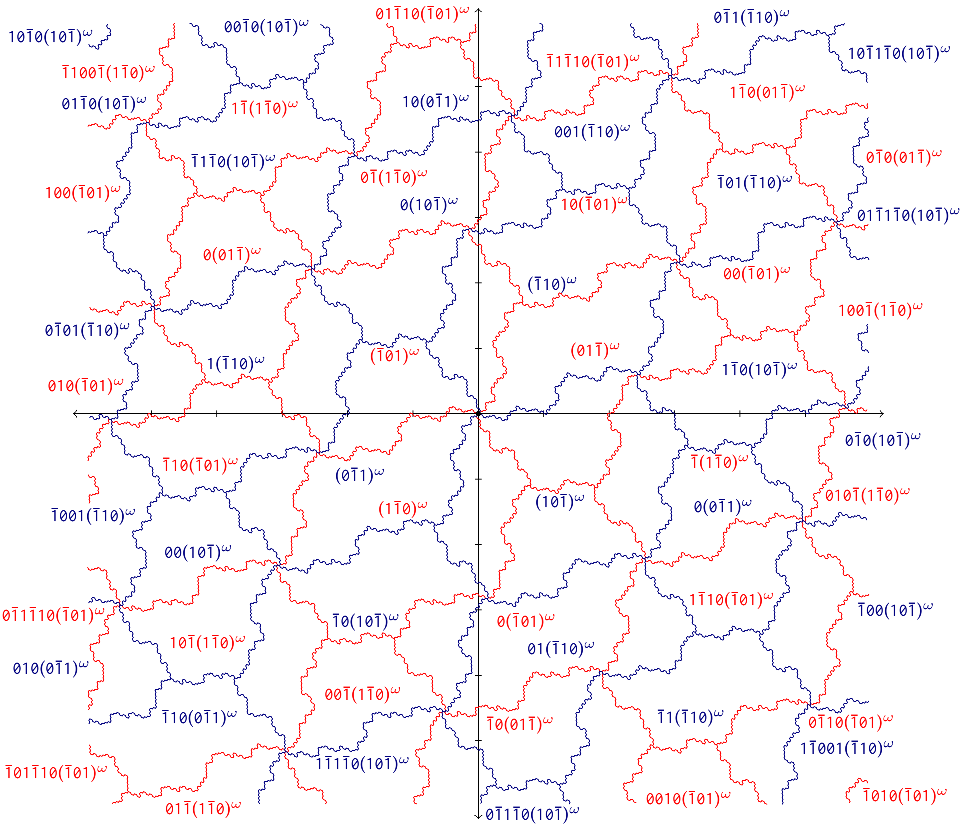
Then $s_i \in (l-1, l+1)$ for all i. We have that $s_0 = x - x = 0$ and also, $s_{i+1} = \beta s_i + x_i - y_i$. Denoting this transition $s_i \xrightarrow{x_i|y_i} s_{i+1}$, we can construct a transducer. It is not difficult to see that only the states 0 and $\bullet 1^n$ with $1 \leqslant n \leqslant d$ are reachable from $s_0 = 0$, simply by verifying that any other transition that the ones listed below would lead to a state which is outside of the interval $(l-1, l+1)$. The transitions are the following:

$$0 \xrightarrow{0|0} 0, \quad 0 \xrightarrow{1|1} 0, \quad 0 \xrightarrow{1|0} \bullet 1^d, \quad \bullet 1 \xrightarrow{0|0} \bullet 1^d, \quad \bullet 1 \xrightarrow{1|1} \bullet 1^d,$$
$$\bullet 1^n \xrightarrow{0|1} \bullet 1^{n-1} \quad \text{for } 1 \leqslant n \leqslant d.$$

The transducer is depicted in Figure 3-2 for the case $d = 3$. When $i \geqslant k$, we get that $x_{i+1} x_{i+2} \cdots = 0^\omega$. The transducer contains only two cycles which read 0^ω on the input (see the thick arrows in Figure 3-2). First, it is the loop $0 \xrightarrow{0|0} 0$; however, it outputs 0^ω, which is forbidden as a balanced expansion. Second, it is the cycle $\bullet 1^d \xrightarrow{0|1} \bullet 1^{d-1} \xrightarrow{0|1} \cdots \xrightarrow{0|1} \bullet 1 \xrightarrow{0|0} \bullet 1^d$. This cycle has the string $(1^{d-1}0)^\omega$ as an output; therefore every balanced expansion of $x \in \mathbb{Z}[\beta] \cap [l, 1)$ has $(1^{d-1}0)^\omega$ as a suffix.

For numbers $x \in \mathbb{Z}[\beta] \cap (1, l+1)$ we observe that $T_B^n(x) < 1$ for some $n \in \mathbb{N}$, therefore the balanced expansions of these numbers have $(1^{d-1}0)^\omega$ as a suffix as well.

We conclude that the purely periodic points in $\mathbb{Z}[\beta]$ are $\bullet(1^{d-1}0)^\omega$ and points in its orbit, which is the claim of the lemma. ☐



(Fig. 3-3) The multiple tiling for the symmetric β-transformation with $\beta = \varphi_t$ the Tribonacci constant.

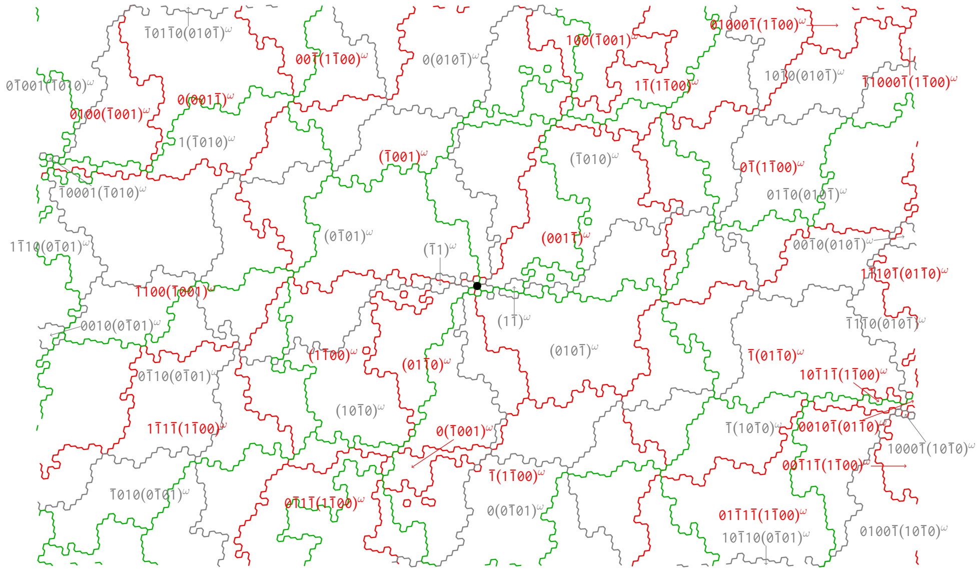
Proof of Proposition 3-7. To prove that T_B induces a tiling (i.e., that $m_B = 1$), it suffices to find $z \in \mathbb{Z}[\beta]$ that lies in only one tile. To this end, let $z = \beta^d + 1 = 10^{d-1}1_\bullet$. Put $k := 4d$; we will see that this choice of k satisfies the hypothesis of Lemma 3-8 for all $y \in \mathcal{P}_B$. Let $y = {}_\bullet 1^n$ with $1 \leqslant n \leqslant d$. Then, because $10^d{}_\bullet = 01^d{}_\bullet$, we get that

$$\begin{aligned} y + \beta^{-4d} z &= {}_\bullet 1^n 0^{d-n} \quad 0^d \quad 0^{d-1}1 \quad 0^{d-1}1 \quad 0^\omega \\ &= {}_\bullet 1^{n-1} 0 1^{d-n} 1^n 0^{d-n} \quad 0^{d-1}1 \quad 0^{d-1}1 \quad 0^\omega \\ &= \begin{cases} {}_\bullet 1^{n-1}01^{d-n}1^{n-1}01^{d-n}1^n0^{d-n-1}10^d \quad (1^{d-1}0)^\omega & \text{if } n \leqslant d-2, \\ {}_\bullet 1^{d-2}01 \quad 1^{d-1}0 \quad 0^{d-1}1 \quad 0^d \quad (1^{d-1}0)^\omega & \text{if } n = d-1, \\ {}_\bullet 1^{d-1}0 \quad 1^d \quad 0^{d-1}1 \quad 0^d \quad (1^{d-1}0)^\omega & \text{if } n = d. \end{cases} \end{aligned}$$

None of the strings we derived contains 0^{d+1} nor 1^{d+1} as a factor, therefore they are the balanced expansions of $y + \beta^{-4d}z$ by Lemma 3-9. Also, we know that $y = {}_\bullet(1^{n-1}01^{d-n})^\omega$ and the strings have $1^{n-1}01^{d-n}$ as a prefix, so the hypothesis of Lemma 3-8 is satisfied. From this we get that

$$T_B^{4d}(y + \beta^{-4d}z) = {}_\bullet(1^{d-1}0)^\omega = 1 \quad \text{for all } y \in \mathcal{P}_B,$$

whence z lies in only one tile, namely $\mathcal{R}_B(1)$. $\square$



(Fig. 3-4) A cross section through the multiple tiling for the symmetric β-transformation with β the 4-Bonacci constant.

Proof of Theorem 3-2. We know from Proposition 3-7 that $m_B = 1$, and from (3-5) that $|N(\beta - 1)| = d - 1$. The statement then follows from Theorem 3-1. $\square$

3-5 The minimal Pisot case

For all the d-Bonacci numbers, T_B induces a single tiling. However, for $\beta = \varphi_p$, the minimal Pisot number, we know from [**KS12**, § 4.5.2] that T_S induces a double tiling, i.e., that $m_S = 2$; this tiling is depicted in Figure 3-5. Together with the fact that $\beta - 1 = \beta^{-4}$, whence $\beta - 1$ is a unit and $|N(\beta - 1)| = 1$, we get that T_B induces a double tiling.

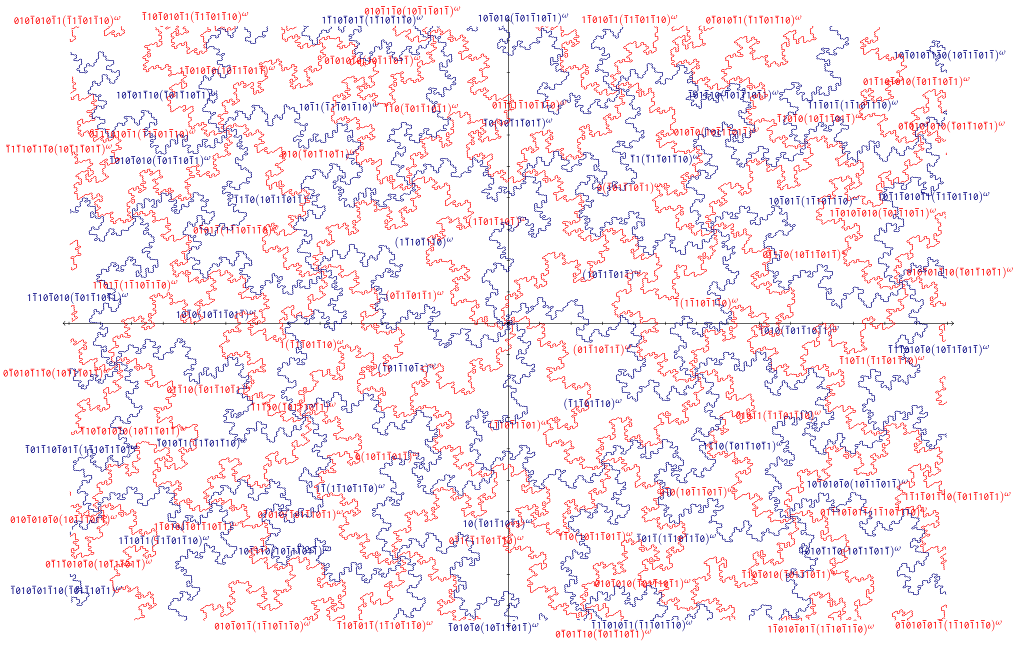
3-6 Continuation of the work

We finish this chapter with several open questions:

Problem 3a. Is there any $c_0 \in (\varphi_p, 2)$ such that the balanced expansions induce a single tiling for all Pisot units $\beta \in (c_0, 2)$?

Problem 3b. What is the degree of the multiple tiling for the symmetric expansions for the (d, a)-Bonacci numbers, i.e., roots of $X^d = aX^{d-1} + aX^{d-2} + \cdots + aX + a$ for $d \geqslant 2$ and $a \geqslant 2$?

Problem 3c. Consider the transformation $T_{\beta,l} \colon [l, l+1) \to [l, l+1)$, $x \mapsto \beta x - \lfloor \beta x + l \rfloor$, for a d-Bonacci number β. We know that $T_{\beta,0}$ induces a single tiling, because β has Property (F) [**FS92**]. We know from this article that $T_{\beta,-1/2}$



(Fig. 3-5) The multiple tiling for the symmetric β-transformation with $\beta = \varphi_p$ the minimal Pisot number.

induces a multiple tiling of covering degree $d-1$. What happens if $-\frac{1}{2} < l < 0$? What are the possible values of the degree?

Problem 3d. What can we say about the symmetric transformation for $\beta > 2$? Is it possible that it always induces a single tiling?

CHAPTER FOUR

Purely periodic expansions

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4-1 Introduction

In this chapter, we are going to investigate an interesting property of the Greedy β-expansions. We recall that they are given by the transformation

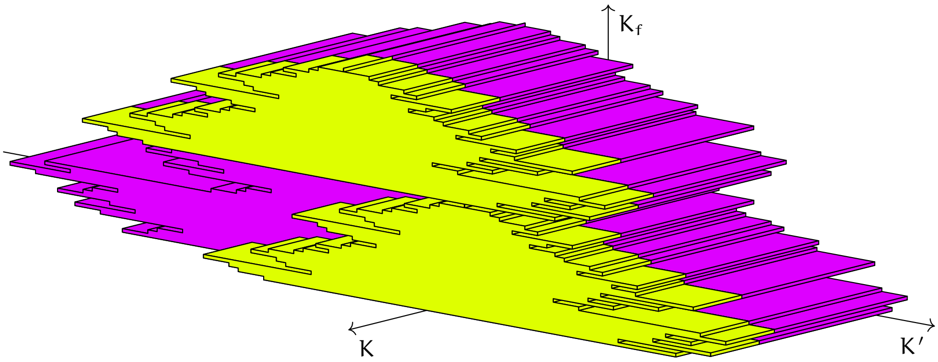
$$T_G\colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor.$$

For $\beta \in \mathbb{N}$, we recover the standard expansions in base β and the β-expansion of $x \in [0,1)$ is eventually periodic (i.e., there exist p, n such that $x_{k+p} = x_k$ for all $k \geqslant n$) if and only if $x \in \mathbb{Q}$. This result was generalized to all Pisot bases by Schmidt [**Sch80**], who proved that for a Pisot number β the expansion of $x \in [0,1)$ is eventually periodic if and only if x is an element of the number field $\mathbb{Q}(\beta)$. Moreover, he showed that when β satisfies $\beta^2 = a\beta + 1$, then each $x \in [0,1) \cap \mathbb{Q}$ has a purely periodic β-expansion.

Akiyama [**Aki98**] showed that if β is a Pisot unit satisfying Property (F) then there exists $c > 0$ such that all rational numbers $x \in \mathbb{Q} \cap [0, c)$ have a purely periodic expansion. If β is not a unit, then a rational number $p/q \in [0,1)$ can have a purely periodic expansion only if q is co-prime to the norm $N(\beta)$. As an example, we know that the expansion of $1/6$ in base 2 is $\bullet 0010101\cdots$, which is not purely periodic. Many Pisot non-units satisfy that there exists $c > 0$ such that all rational numbers $\frac{p}{q} \in [0, c)$ with q co-prime to b have a purely periodic expansion. This stimulates for the following definition:

Definition 4-1. Let β be a Pisot number, and let $N(\beta)$ denote the norm of β. Then we define $\gamma(\beta) \in [0,1]$ as the maximal c such that all $\frac{p}{q} \in \mathbb{Q} \cap [0, c)$ with $\gcd(q, N(\beta)) = 1$ have a purely periodic β-expansion. In other words,

$$\gamma(\beta) := \inf\{\tfrac{p}{q} \in \mathbb{Q} \cap [0,1) : \gcd(q, N(\beta)) = 1,\\ \tfrac{p}{q} \text{ has a not purely periodic greedy } \beta\text{-expansion}\} \cup \{1\}.$$



(Fig. 4-1) The natural extension domain for $\beta = 1 + \sqrt{3}$.

The question is how to determine the value of $\gamma(\beta)$. As well, knowing when $\gamma(\beta) = 0$ or 1 is of big interest. Values of $\gamma(\beta)$ for whole classes of numbers as well as for particular numbers have been given [**Aki98**, **ABBS08**, **AS05**, **MS14**, **Sch80**]. Generic results on the cubic case were obtained [**AFSS10**] stating that when β is a cubic number, then $\gamma(\beta) > 0$ if and only if β is a Pisot unit with Property (F). Periodic quadratic unit Ito-Sadahiro (negative base) expansions were studied [**MP13**].

It is easy to observe that the expansion of x is purely periodic if and only if x is a periodic point of T_G, i.e., there exists $p \geqslant 1$ such that $T_G^p x = x$. The natural extension $(\mathcal{X}, \mathcal{T})$ of the dynamical system $([0, 1), T_G)$ can be defined in an algebraic way, cf. § 2-3-1. An example of the natural extension domain is in Figure 4-1. Several authors contributed to proving the following result: A point $x \in [0, 1)$ has a purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and its diagonal embedding lies in the natural extension domain $\mathcal{X}$. The quadratic unit case was solved by M. Hama and T. Imahashi [**HI97**], the confluent unit case by S. Ito and Y. Sano [**IS01**, **IS02**]. Then S. Ito and H. Rao [**IR05**] resolved the unit case completely using an algebraic argument. For non-unit bases β, one has to consider finite (p-adic) places of the field $\mathbb{Q}(\beta)$. This consideration allowed V. Berthé and A. Siegel [**BS07**] to expand the result to all (non-unit) Pisot numbers.

The first values of $\gamma(\beta)$ for two particular quadratic non-units were provided by Akiyama et al. [**ABBS08**]. Recently, Minervino and Steiner [**MS14**] described the boundary of $\mathcal{X}$ for quadratic non-unit Pisot bases. This allowed them to find the value of $\gamma(\beta)$ for an infinite class of quadratic numbers:

Theorem 4-2 [MS14]. *Let β be the positive root of $\beta^2 = a\beta + b$ for $a \geqslant b \geqslant 1$ two co-prime integers. Then*

$$\gamma(\beta) = \begin{cases} 1 - \frac{(b-1)b\beta}{\beta^2 - b^2} & \text{if } a > b(b-1), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\gamma(\beta) = 1$ if and only if $b = 1$.

This chapter is organized as follows: In the next section, the main results are stated. In § 4-4, properties of β-adic expansions are studied. Section 4-5 connects tiles arising from the β-transformation and the value $\gamma(\beta)$ in order to prove Theorem 4-4. The proof of Theorem 4-5 is completed in § 4-6, together with that of Theorem 4-6. Comments on the general case are in § 4-7. A list of related open questions closes the chapter.

4-2 Main results

The purpose of this chapter is to generalize Theorem 4-2 to all quadratic Pisot numbers with norm $N(\beta) < 0$. (Note that when $N(\beta) > 0$, then β has a positive Galois conjugate $\beta' > 0$ and $\gamma(\beta) = 0$ [**Aki98**, Proposition 5].) To this end, we define β-adic expansions (not to be confused with the Rényi β-expansions) similarly to p-adic expansions with $p \in \mathbb{Z}$, see also § 4-4.

Definition 4-3. Let β be an algebraic integer. The *β-adic expansion* of $x \in \mathbb{Z}[\beta]$ is the unique infinite word $\mathbf{h}(x) := u_0 u_1 u_2 \cdots$ such that $u_n \in \{0, 1, \dots, |N(\beta)| - 1\}$ and $x - \sum_{i=0}^{n-1} u_i \beta^i \in \beta^n \mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$.

Theorem 4-4. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a \geqslant b \geqslant 1$. Then*

$$\gamma(\beta) = \begin{cases} 0 & \text{if } \sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j-\beta); \beta' \rangle > \beta \\ & \quad \text{or } \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle < -1, \\ \beta - a & \text{if } \sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j-\beta); \beta' \rangle \in (2\beta - a - 1, \beta] \\ & \quad \text{and } \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle \geqslant \beta - a - 1, \\ 1 + \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle & \text{otherwise,} \end{cases}$$

where $\langle u_0 u_1 u_2 \cdots ; X \rangle := \sum_{n\geqslant 0} u_n X^n$.

In many cases, we obtain the following direct formula (which we conjecture to be true for all $a \geqslant b \geqslant 1$):

Theorem 4-5. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ for $a \geqslant b \geqslant 1$. Suppose $a > \frac{1+\sqrt{5}}{2} b$ or $a = b$ or $\gcd(a, b) = 1$. Then*

$$\gamma(\beta) = \max\{0, 1 + \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle\}. \tag{4-1}$$

The infimum in (4-1) can be easily computed with the help of Proposition 4-11 below. In the case $\frac{a}{b} \in \mathbb{Z}$, Proposition 4-13 provides an even faster algorithm, and we are able to prove a necessary and sufficient condition for $\gamma(\beta) = 1$:

Theorem 4-6. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a \geqslant b \geqslant 1$ and such that b divides a.*

1. *We have that $\gamma(\beta) = 1$ if and only if $a \geqslant b^2$ or $(a, b) \in \{(24, 6), (30, 6)\}$.*
2. *If $a = b \geqslant 3$ then $\gamma(\beta) = 0$.*

4-3 Notation

For the purposes of this chapter, we fix some convenient notation. We denote by $\mathrm{Pref}_n\,\mathbf{u}$ the prefix of length n of an infinite word $\mathbf{u}$.

To a finite word $w = w_0 w_1 \dots w_{k-1}$ we assign the polynomial

$$\langle w; X\rangle := \sum_{i=0}^{k-1} w_i X^i.$$

Similarly, $\langle \mathbf{u}; X\rangle := \sum_{i\geqslant 0} u_i X^i$ is a formal power series for an infinite word $\mathbf{u} = u_0 u_1 u_2 \cdots$.

For integers $a, b \in \mathbb{Z}$, we denote by $a \perp b$ the fact that a and b are co-prime, i.e., that $\gcd(a, b) = 1$. Moreover, for $b \geqslant 2$ we put $\mathbb{Z}_b := \{\, p/q : p, q \in \mathbb{Z},\ q \perp b \,\}$ (the ring of rational numbers with denominator co-prime to b).

Since we consider only quadratic non-unit β, we get that $K := \mathbb{Q}(\beta)$ admits a unique non-identical Galois isomorphism, which we denote $x \mapsto x'$. We also have the representation spaces as introduced in § 2-1-2, and we put

$$S_f := \overline{\{\, x_f : x \in S \,\}} \quad \text{for any } S \subseteq K.$$

In particular, we consider $\mathbb{Z}[\beta]_f$, which is a compact subset of K_f. Since multiplication by β_f is a contraction on K_f, we have that $\beta_f^n \mathbb{Z}[\beta]_f \to \{0_f\}$ as $n \to \infty$.

We consider only quadratic Pisot numbers, whence there is only one non-identical Galois isomorphism, we denote $x \mapsto x'$, and we denote K' accordingly.

4-4 Beta-adic expansions

In Definition 4-3, β-adic expansions are defined on $\mathbb{Z}[\beta]$. By Lemma 4-8 below, we extend this definition to the closure $\mathbb{Z}[\beta]_f$ similarly to the p-adic case. To this end, let

$$H\colon \mathbb{Z}[\beta]_f \to \mathbb{Z}[\beta]_f, \quad x \mapsto \beta_f^{-1}\big(x - d(x)_f\big),$$

where $d(x)$ is the unique digit $d \in \mathcal{A} := \{0, 1, \dots, |N(\beta)| - 1\}$ such that $\beta_f^{-1}(x - d_f)$ is in $\mathbb{Z}[\beta]_f$. Such d exists because $\mathbb{Z}[\beta] = \mathcal{A} + \beta\mathbb{Z}[\beta]$. It is unique because $(c + \beta\mathbb{Z}[\beta])_f \cap (d + \beta\mathbb{Z}[\beta])_f \neq \emptyset$ implies $(\beta^{-1}(c - d))_f \in \mathbb{Z}[\beta]_f$ and thus $c \equiv d \pmod{N(\beta)}$ by the following lemma:

Lemma 4-7 [MS14, Lemma 5.2 and Eq. (5.1)]. *For each $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$ we have $x_f \notin \mathbb{Z}[\beta]_f$. There exists $k \in \mathbb{N}$ such that $\mathbb{Z}[\beta^{-1}] \cap \beta^k \mathcal{O}_K \subseteq \mathbb{Z}[\beta]$.*

Lemma 4-8. *The β-adic expansion map $\mathbf{h}_f : \mathbb{Z}[\beta]_f \to \mathcal{A}^\omega$ defined by*

$$\mathbf{h}_f(z) := u_0 u_1 u_2 \cdots, \quad \textit{where} \quad u_i := d\big(H^i(z)\big),$$

is a homeomorphism. It satisfies that $\mathbf{h}_f(x_f) = \mathbf{h}(x)$ for all $x \in \mathbb{Z}[\beta]$.

Proof. The map $\mathbf{h}_f$ is surjective because $\mathbf{h}_f(\langle \mathbf{u}; \beta_f \rangle) = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{A}^\omega$. It is injective because $\mathbf{h}_f(z) = \mathbf{u} = u_0 u_1 u_2 \cdots$ implies that $z \in \sum_{i=0}^{n-1} u_i \beta_f^i + \beta_f^n \mathbb{Z}[\beta]_f$ for all n, thus $z = \langle \mathbf{u}; \beta_f \rangle$.

Since $(\mathcal{O}_K)_f$ is open and $\mathbb{Z}[\beta^{-1}]_f = K_f$, we get from Lemma 4-7 that $\mathbb{Z}[\beta]_f = \bigcup_{x \in \mathbb{Z}[\beta]} x_f + \beta_f^k (\mathcal{O}_K)_f$ for some $k \in \mathbb{N}$, and therefore it is an open set as well. Then the pre-image $\mathbf{h}_f^{-1}(w\mathcal{A}^\omega) = \langle w; \beta_f \rangle + \beta_f^n \mathbb{Z}[\beta]_f$ is open for any $w \in \mathcal{A}^*$. As the cylinders $\{ w\mathcal{A}^\omega : w \in \mathcal{A}^* \}$ form a base of the topology of $\mathcal{A}^\omega$, the map $\mathbf{h}_f$ is continuous.

The inverse $\mathbf{h}_f^{-1}$ is continuous because $\beta_f^n \mathbb{Z}[\beta]_f \to \{0_f\}$ as $n \to \infty$.

For $x \in \mathbb{Z}[\beta]$, the equality $\mathbf{h}_f(x_f) = \mathbf{h}(x)$ follows from the fact that $\beta^{-1}(x - d(x_f)) \in \mathbb{Z}[\beta]$. □

Note that we can identify $\mathbb{Z}[\beta]_f$ with the inverse limit space $\varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta]$. Indeed defining the map $\kappa \colon \mathcal{A}^\omega \to \varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta]$,

$$u_0 u_1 u_2 \cdots \mapsto (\xi_1, \xi_2, \xi_3, \dots), \quad \text{where } \xi_n = \sum_{i=0}^{n-1} u_i \beta^i,$$

the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}[\beta]_f \arrow[r, "H"] \arrow[d, "\cong"', "\mathbf{h}"] & \mathbb{Z}[\beta]_f \arrow[d, "\cong"', "\mathbf{h}"] \\
\mathcal{A}^\omega \arrow[r, "\text{(shift)}"] \arrow[d, "\cong"', "\kappa"] & \mathcal{A}^\omega \arrow[d, "\cong"', "\kappa"] \\
\varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta] \arrow[r] & \varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta].
\end{tikzcd}$$

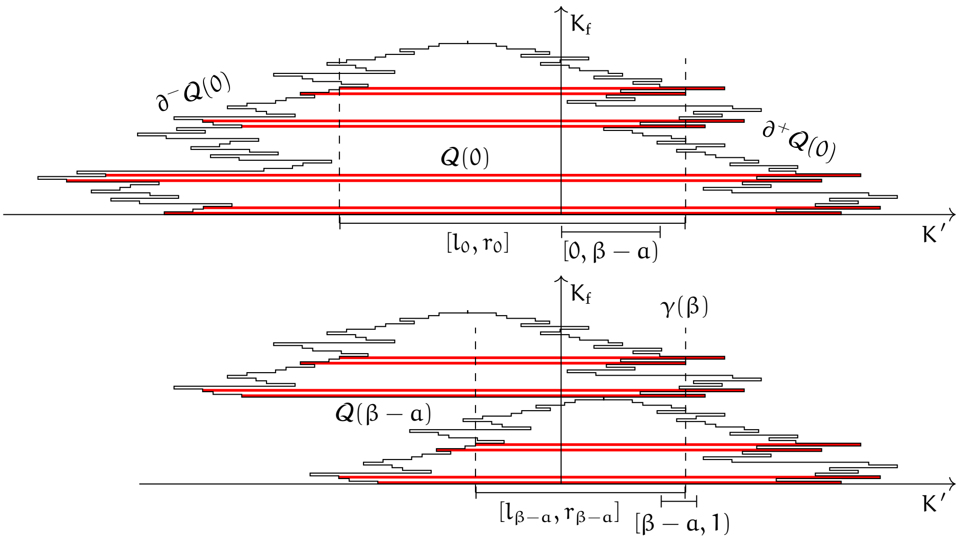
We recall that an *inverse limit space* $\varprojlim X_n$ is given by a sequence of sets $(X_1, X_2, X_3, \cdots)$ and by maps $f_n \colon X_{n+1} \to X_n$, and it is the space of sequences $(\xi_1, \xi_2, \xi_3, \dots)$ such that $\xi_n \in X_n$ and $f_n(\xi_{n+1}) = \xi_n$ for all n. In the case of the inverse limit space $\varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta]$, we have that $\beta^{n+1}\mathbb{Z}[\beta]$ is a sub-ring of $\beta^n\mathbb{Z}[\beta]$, therefore $\mathbb{Z}[\beta]/\beta^{n+1}\mathbb{Z}[\beta]$ embeds naturally into $\beta^n\mathbb{Z}[\beta]$ and f_n is this embedding.

4-5 Rauzy fractals and the value $\gamma(\beta)$

The connection between the Rauzy fractals and the value $\gamma(\beta)$ is established by the following theorem:

Theorem 4-9 [HI97, IR05, BS07]. *For a Pisot number β, we have that x has a purely periodic greedy β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and $\Psi_{0,f}(x) \in \mathcal{X}$.*

We recall that $\Psi_{0,f}$ is the embedding of $\mathbb{Q}(\beta)$ into $\mathbb{R}^d \times K_f$ and $\mathcal{X}$ is the domain of the natural extension of $(T_G, [0,1))$ as given in § 2-3-1.



(Fig. 4-2) The tiles $\mathcal{Q}_f(0)$ and $\mathcal{Q}_f(\beta - a)$ for $\beta = 1 + \sqrt{3}$. The (red) stripes illustrate the intersection of $Y = K' \times (\mathbb{Z})_f$ with the tiles.

The goal of this section is to prove Theorems 4-4 and 4-5, using the connection between β-tiles and the value of $\gamma(\beta)$. First we prove the following lemma about the closures of $\mathbb{Z}$ and $\mathbb{Z}_b$ in K_f:

Lemma 4-10. *We have that* $(\mathbb{Z})_f = (\mathbb{Z}_b)_f = (\mathbb{Z}_b \cap [l, r])_f$ *for all* $l < r$.

Proof. We have that $(\mathbb{Z}_b)_f = (\mathbb{Z}_b \cap [l, r])_f$ by [**ABBS08**, Lemma 4.7]. Clearly $\mathbb{Z} \subseteq \mathbb{Z}_b$ whence $(\mathbb{Z})_f \subseteq (\mathbb{Z}_b)_f$. We will prove that $(\mathbb{Z}_b)_f \subseteq (\mathbb{Z})_f$, namely that every point $x/q \in \mathbb{Z}_b$ for $x, q \in \mathbb{Z}$ and $q \perp b$ can be approximated by integers. For each $n \in \mathbb{N}$, there exists $q_n \in \mathbb{Z}$ such that $q_n q \equiv 1 \pmod{b^n}$. Then $\frac{x}{q} - q_n x = (1 - q_n q)\frac{x}{q} \in \frac{1}{q} b^n \mathbb{Z} \subseteq \frac{1}{q}\beta^n \mathbb{Z}[\beta]$, therefore $(q_n x)_f \to (x/q)_f$. □

Proof of Theorem 4-4. By Definition 4-1, Theorem 4-9 and since $(1, \Psi_f(1)) \notin \mathcal{X}$, we have that

$$\gamma(\beta) = \inf\{\, x \in \mathbb{Z}_b : x \geqslant 0,\ \Psi_{0,f}(x) \notin \mathcal{X} \,\}.$$

For $x \in \mathbb{Q} \cap [0, \beta - a)$, the condition $\Psi_{0,f}(x) \in \mathcal{X}$ is equivalent to $\Psi_f(x) \in \mathcal{Q}_f(0)$; for $x \in \mathbb{Q} \cap [\beta - a, 1)$, it is equivalent to $\Psi_f(x) \in \mathcal{Q}_f(\beta - a)$.

We recall the results of [**MS14**, § 9.3], where the shape of the tiles is described. The intersection of $\mathcal{Q}_f(x)$ with a line $K' \times \{z\}$ is a line segment for any $z \in \mathbb{Z}[\beta]_f$ and it is empty for all $z \in K_f \setminus \mathbb{Z}[\beta]_f$, see Figure 4-2. Let $\partial^- \mathcal{Q}_f(x)$ denote the set of the segments' left end-points, and similarly $\partial^+ \mathcal{Q}_f(x)$ the set of the right end-points. Put

$$l_x := \sup \pi'(\partial^- \mathcal{Q}_f(x) \cap Y) \quad \text{and} \quad r_x := \inf \pi'(\partial^+ \mathcal{Q}_f(x) \cap Y) \quad \text{for } x = 0, \beta - a,$$

where $Y := K' \times (\mathbb{Z}_b)_f \subseteq K' \times \mathbb{Z}[\beta]_f$ and π' denotes the projection $\pi' \colon K' \times K_f \to K'$, $(y, z) \mapsto y$. Then all numbers $p/q \in \mathbb{Z}_b$ in $[l_0, r_0] \cap [0, \beta - a)$ have a purely

$$\begin{tikzcd}
\boxed{\beta-a} \arrow[r, "{0,1,\dots,b-1}"] & \boxed{a+1-\beta} \arrow[r, bend left, "{a-b+1,\dots,a}"] & \boxed{\beta-a-1} \arrow[l, bend left, "{0,1,\dots,b-1}"] & \boxed{a-\beta} \arrow[l, "{a-b,\dots,a-1}"']
\end{tikzcd}$$

(Fig. 4-3) Boundary graph for quadratic β-tiles, cf. [**MS14**, Fig. 6]. Each arrow in the graph represents exactly b edges.

periodic expansion, and so do all numbers $p/q \in \mathbb{Z}_b$ in $[l_{\beta-a}, r_{\beta-a}] \cap [\beta - a, 1)$. Outside these two sets, numbers $p/q \in \mathbb{Z}_b$ that do not have a purely periodic expansion are dense, since the points $\Psi_f(p/q)$ are dense in Y by Lemma 4-10. Therefore, the value of $\gamma(\beta)$ depends on the relative position of the above intervals (see Figure 4-2) in the following way:

$$\gamma(\beta) = \begin{cases} 0 & \text{if } l_0 > 0 \text{ or } r_0 < 0, \\ r_0 & \text{if } l_0 \leqslant 0 \text{ and } r_0 \in [0, \beta - a), \\ \beta - a & \text{if } l_0 \leqslant 0,\, r_0 \geqslant \beta - a \text{ and } \beta - a \notin [l_{\beta-a}, r_{\beta-a}], \\ \min\{r_{\beta-a}, 1\} & \text{if } l_0 \leqslant 0,\, r_0 \geqslant \beta - a \text{ and } \beta - a \in [l_{\beta-a}, r_{\beta-a}]. \end{cases} \tag{4-2}$$

In the rest of the proof, we will show that

$$l_0 = l_{\beta-a} - 1 = -\beta + \sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j - \beta); \beta' \rangle, \tag{4-3}$$

$$r_0 = r_{\beta-a} = 1 + \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle. \tag{4-4}$$

As $\inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle \leqslant \langle \mathbf{h}(0); \beta' \rangle = 0$, we see that (4-2) implies the statement of the theorem.

We use results of [**MS14**, §§ 8.3, 9.2 and 9.3], namely Equations (8.4) and (9.2), which read:

$$z \in \mathcal{R}_f(x) \cap \mathcal{R}_f(y) \quad \text{if and only if} \quad z = \Psi_f(x) + \langle \mathbf{u}; \Psi_f(\beta) \rangle,$$

where $\mathbf{u} = v_0 v_1 v_2 \cdots$ is an edge-labelling of a path in the boundary graph in Figure 4-3 that starts in the node $y - x$; and

$$\partial\mathcal{R}_f(x) = \Big(\mathcal{R}_f(x) \cap \mathcal{R}_f\big(x + \beta - \lfloor x + \beta \rfloor\big)\Big) \cup \Big(\mathcal{R}_f(x) \cap \mathcal{R}_f\big(x - \beta - \lfloor x - \beta \rfloor\big)\Big),$$

where the first part is the left boundary $\mathcal{R}_f^-(x)$ and the second part is the right boundary $\mathcal{R}_f^+(x)$. Therefore

$$\partial^-\mathcal{R}_f(0) = \partial^+\mathcal{R}_f(\beta - a) = \{\, \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in (\mathcal{A}\mathcal{B})^\omega \,\},$$
$$\partial^+\mathcal{R}_f(0) = \{\, \Psi_f(a + 1 - \beta) + \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in (\mathcal{A}\mathcal{B})^\omega \,\},$$
$$\partial^-\mathcal{R}_f(\beta - a) = \{\, \Psi_f(\beta - a) + \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in (\mathcal{A}\mathcal{B})^\omega \,\},$$

where we put $\mathcal{B} \coloneqq \{a{-}b{+}1, a{-}b{+}2, \dots, a\}$. We have that

$$\begin{aligned} \{\, \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in (\mathcal{A}\mathcal{B})^\omega \,\} &= \{\, \big\langle ((b-1)a)^\omega; \Psi_f(\beta) \big\rangle - \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in \mathcal{A}^\omega \,\} \\ &= -\Psi_f(1) - \{\, \langle \mathbf{u}; \Psi_f(\beta) \rangle : \mathbf{u} \in \mathcal{A}^\omega \,\}, \end{aligned}$$

since $\mathcal{A} = b - 1 - \mathcal{A}$ and $\mathcal{B} = a - \mathcal{A}$. Because $\mathcal{Q}_f(x) = \Psi_f(x) - \mathcal{R}_f(x)$, we have $\partial^{\pm}\mathcal{Q}_f(x) = \Psi_f(x) - \partial^{\mp}\mathcal{R}_f(x)$. We obtain

$$\begin{aligned} \partial^{-}\mathcal{Q}_f(0) &= \Psi_f(\beta - a) + \{\, \langle \mathbf{u}; \Psi_f(\beta)\rangle : \mathbf{u} \in \mathcal{A}^{\omega} \,\}, \\ \partial^{-}\mathcal{Q}_f(\beta - a) &= \Psi_f(\beta - a + 1) + \{\, \langle \mathbf{u}; \Psi_f(\beta)\rangle : \mathbf{u} \in \mathcal{A}^{\omega} \,\}, \\ \partial^{+}\mathcal{Q}_f(0) = \partial^{+}\mathcal{Q}_f(\beta - a) &= \Psi_f(1) + \{\, \langle \mathbf{u}; \Psi_f(\beta)\rangle : \mathbf{u} \in \mathcal{A}^{\omega} \,\}. \end{aligned}$$

We have that

$$\begin{aligned} \Psi_f(1) + \langle \mathbf{u}; \Psi_f(\beta)\rangle \in Y \iff 1_f + \langle \mathbf{u}; \beta_f\rangle \in \mathbb{Z}_f \\ \iff \langle \mathbf{u}; \beta_f\rangle \in \mathbb{Z}_f \iff \mathbf{u} \in \mathbf{h}_f(\mathbb{Z}_f), \end{aligned}$$

because $\mathbf{h}_f(\langle \mathbf{u}; \beta_f\rangle) = \mathbf{u}$ and $\mathbf{h}_f$ is a homeomorphism by Lemma 4-8. Then, since the map $\mathbb{Z}_f \to K'$, $z \mapsto \langle \mathbf{h}_f(z); \beta'\rangle$ is continuous, we get that

$$\inf \pi'\big(\partial^{+}\mathcal{Q}_f(x) \cap Y\big) = 1 + \inf_{z\in\mathbb{Z}_f} \langle \mathbf{h}_f(z); \beta'\rangle = 1 + \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta'\rangle,$$

which justifies (4-4). Similarly, $\Psi_f(\beta - a) + \langle \mathbf{u}; \Psi_f(\beta)\rangle \in Y$ if and only if $\mathbf{u} \in \mathbf{h}_f(\mathbb{Z}_f - \beta_f)$, therefore

$$\sup \pi'\big(\partial^{-}\mathcal{Q}_f(\beta - a) \cap Y\big) - 1 = \sup \pi'\big(\partial^{-}\mathcal{Q}_f(0) \cap Y\big) = \beta' - a + \sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j - \beta); \beta'\rangle.$$

Since $\beta' - a = -\beta$, this justifies (4-3). □

Proof of Theorem 4-5, case $a > \frac{1+\sqrt{5}}{2} b$. Since $\beta' < 0$, we have that

$$\sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j - \beta); \beta'\rangle \leqslant \sup_{\mathbf{u}\in\mathcal{A}^{\omega}} \langle \mathbf{u}; \beta'\rangle = \big\langle ((b{-}1)0)^{\omega}; \beta'\big\rangle = \frac{b - 1}{1 - (\beta')^2}.$$

We will show that this quantity is $< 2\beta - a - 1$. First, we derive, using $(\beta')^2 = a\beta' + b$, $\beta = a - \beta'$ and $1 - (\beta')^2 > 0$, that it is equivalent to

$$a + ab + \beta'(a^2 + a + 2b - 2) > 0. \tag{4-5}$$

We know that $\beta < a + 1$, therefore $\beta = a + \frac{b}{\beta} > \frac{a(a+1)+b}{a+1}$ and $\beta' = -\frac{b}{\beta} > -\frac{(a+1)b}{a^2+a+b}$. As well, $a^2 + a + 2b - 2 > 0$, therefore we estimate

$$a + ab + \beta'(a^2 + a + 2b - 2) > \frac{ab^2((\frac{a}{b})^2 - \frac{a}{b} - 1) + b^2((\frac{a}{b})^2 + 2\frac{a}{b} - 2) + 2b}{a^2 + a + b}.$$

When $\frac{a}{b} > \frac{1+\sqrt{5}}{2}$, all three terms in the numerator are positive. Since the denominator is also positive, we get that $\sup_{j\in\mathbb{Z}} \langle \mathbf{h}(j - \beta); \beta'\rangle < 2\beta - a - 1$. Theorem 4-4 then implies (4-1). □

TABLE 4-1
The values of $\gamma(\beta)$ for the case when b divides a. The star '$\star$' means that the value is strictly between 0 and 1.

$a/b =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b = 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1
4	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
5	0	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1
6	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
7	0	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1
8	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1
9	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1
10	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1
11	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1
12	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1

The proof of the case $a \perp b$ of Theorem 4-5 was given in [**MS14**, § 9]. The proof of the case $a = b$ is given in the next section on page 40, because it falls under the case when b divides a.

The following proposition shows how to compute the infimum in Theorem 4-5 and thus the value of $\gamma(\beta)$ in a lot of (and possibly all) cases. Comments on the computation of $\gamma(\beta)$ by Theorem 4-4 are in Section 4-7. We recall that $\operatorname{Pref}_n \mathbf{u}$ denotes the prefix of $\mathbf{u}$ of length n.

Proposition 4-11. *Let $\beta^2 = a\beta + b$ with $a \geqslant b \geqslant 2$. Then for each $n \in \mathbb{N}$ we have*

$$\inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle \in \min_{j\in\{0,1,\dots,b^n-1\}} \left\langle \operatorname{Pref}_n \mathbf{h}(j); \beta' \right\rangle + (\beta')^n \tfrac{b-1}{1-(\beta')^2} [\beta', 1]. \tag{4-6}$$

Lemma 4-12. *Let $x, y \in \mathbb{Z}[\beta]$ satisfy that $x - y \in b^n\mathbb{Z}[\beta]$. Then* $\operatorname{Pref}_n \mathbf{h}(x) = \operatorname{Pref}_n \mathbf{h}(y)$.

Proof. Since $b = \beta^2 - a\beta \in \beta\mathbb{Z}[\beta]$, we have that $x - y \in \beta^n\mathbb{Z}[\beta]$. Let $\mathbf{h}(x) = u_0u_1\cdots$. Then $x - \sum_{j=0}^{n-1} u_j\beta^j \in \beta^n\mathbb{Z}[\beta]$ and therefore $y - \sum_{j=0}^{n-1} u_j\beta^j \in \beta^n\mathbb{Z}[\beta]$, which means that $u_0\cdots u_{n-1}$ is a prefix of $\mathbf{h}(y)$. □

Proof of Proposition 4-11. Set $\mu_n \coloneqq \min_{j\in\{0,1,\dots,b^n-1\}} \langle \operatorname{Pref}_n \mathbf{h}(j); \beta' \rangle$. The statement actually consists of two inequalities, which will be proved separately. Let $j \in \mathbb{Z}$. Since $\operatorname{Pref}_n \mathbf{h}(j) = \operatorname{Pref}_n \mathbf{h}(j \bmod b^n)$ by Lemma 4-12 and since $\beta' < 0$, we have

$$\begin{aligned}
\langle \mathbf{h}(j); \beta' \rangle &\geqslant \left\langle \operatorname{Pref}_n \mathbf{h}(j)(0(b-1))^\omega; \beta' \right\rangle \geqslant \mu_n + (\beta')^{n+1} \tfrac{b-1}{1-(\beta')^2} && \text{if } n \text{ is even,}\\
\langle \mathbf{h}(j); \beta' \rangle &\geqslant \left\langle \operatorname{Pref}_n \mathbf{h}(j)((b-1)0)^\omega; \beta' \right\rangle \geqslant \mu_n + (\beta')^{n} \tfrac{b-1}{1-(\beta')^2} && \text{if } n \text{ is odd.}
\end{aligned}$$

TABLE 4-2
Numerical values of $\gamma(\beta)$, where $\beta^2 = a\beta + b$, that correspond to the first several '$\star$' in Table 4-1.

a	b	$\gamma(\beta)$
2	2	$0.914803044196\cdots$
6	3	$0.992963560101\cdots$
8	4	$0.933542944675\cdots$
12	4	$0.999897789000\cdots$
10	5	$0.834150794175\cdots$
15	5	$0.995306723671\cdots$
20	5	$0.999999907110\cdots$
12	6	$0.736114178272\cdots$
18	6	$0.993897266395\cdots$

a	b	$\gamma(\beta)$
14	7	$0.584906533458\cdots$
21	7	$0.944526094618\cdots$
28	7	$0.997984788082\cdots$
35	7	$0.999986041767\cdots$
42	7	$0.999999999997111\cdots$
16	8	$0.351975291826\cdots$
24	8	$0.920692804616\cdots$
32	8	$0.993476100312\cdots$
40	8	$0.999605537625\cdots$
48	8	$0.999999588706\cdots$
56	8	$0.999999999999826\cdots$

To prove the other inequality, let $k \in \{0, \ldots, b^n - 1\}$ be such that $\mu_n = \langle \mathrm{Pref}_n\, \mathbf{h}(k); \beta'\rangle$. Then

$$\begin{aligned} \langle \mathbf{h}(k); \beta'\rangle &\leqslant \big\langle \mathrm{Pref}_n\, \mathbf{h}(k)((b{-}1)0)^{\omega}; \beta'\big\rangle = \mu_n + (\beta')^n \tfrac{b-1}{1-(\beta')^2} && \text{if } n \text{ is even,}\\ \langle \mathbf{h}(k); \beta'\rangle &\leqslant \big\langle \mathrm{Pref}_n\, \mathbf{h}(k)(0(b{-}1))^{\omega}; \beta'\big\rangle = \mu_n + (\beta')^{n+1} \tfrac{b-1}{1-(\beta')^2} && \text{if } n \text{ is odd;} \end{aligned}$$

this provides the upper bound on the infimum. □

4-6 The case b divides a

In this section, we aim to prove Theorem 4-6, which deals with the particular case when b divides a. Table 4-1 shows whether $\gamma(\beta)$ is 0, 1 or strictly in between, for $b \leqslant 12$ and $a/b \leqslant 15$. The first non-trivial values are listed in Table 4-2. The algorithm for obtaining these values is deduced from Theorem 4-5 (which covers all the cases when $\frac{a}{b} \in \mathbb{Z}$ since then either $a = b$ or $a \geqslant 2b > \frac{1+\sqrt{5}}{2} b$), and the following proposition, which improves the statement of Proposition 4-11.

Proposition 4-13. *Let $\beta^2 = a\beta + b$ with $a \geqslant b \geqslant 2$ and $\frac{a}{b} \in \mathbb{Z}$. Then for each $n \in \mathbb{N}$ we have*

$$\inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta'\rangle \in \min_{j\in\{0,1,\ldots,b^n-1\}} \big\langle \mathrm{Pref}_{2n}\, \mathbf{h}(j); \beta'\big\rangle + (\beta')^{2n} \tfrac{b-1}{1-(\beta')^2} [\beta', 0].$$

Lemma 4-14. *Let $\beta^2 = cb\beta + b$. Let $x, y \in \mathbb{Z}[\beta]$ satisfy that $x - y \in b^n\mathbb{Z}[\beta]$ for some $n \in \mathbb{N}$. Then $\mathrm{Pref}_{2n}\, \mathbf{h}(x) = \mathrm{Pref}_{2n}\, \mathbf{h}(y)$. Moreover, for all $x \in \mathbb{Z}[\beta]$ and $d \in \mathcal{A}$ there exists $y \in x + b^n\mathcal{A}$ such that $\mathrm{Pref}_{2n+1}\, \mathbf{h}(y) = \mathrm{Pref}_{2n}\, \mathbf{h}(x)d$.*

Proof. We have $\beta^2 = b(c\beta + 1) \in b\mathbb{Z}[\beta]$ and $b = \beta^2 - c(1 + c^2 b)\beta^3 + c^2\beta^4 \in \beta^2 + \beta^3\mathbb{Z}[\beta] \subseteq \beta^2\mathbb{Z}[\beta]$, whence $\beta^2\mathbb{Z}[\beta] = b\mathbb{Z}[\beta]$ and $\beta^{2n}\mathbb{Z}[\beta] = b^n\mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$. Following the lines of the proof of Lemma 4-12, we obtain that if $x - y \in b^n\mathbb{Z}[\beta]$ then $\mathbf{h}(x)$ and $\mathbf{h}(y)$ have a common prefix of length at least $2n$.

Let $\mathbf{h}(x) = u_0u_1\cdots$ be the Hensel expansion of x. Since $b^n \in \beta^{2n} + \beta^{2n+1}\mathbb{Z}[\beta]$, we have that $u_0u_1\cdots u_{2n-1}d$ is a prefix of $\mathbf{h}(x+eb^n)$ for any $e \equiv d - u_{2n} \pmod{b}$. □

Proof of Proposition 4-13. We follow the lines of the proof of Proposition 4-11 for the case n even. The lower bound is the same in both statements, therefore we only need to prove that $\inf_{j\in\mathbb{Z}}\langle \mathbf{h}(j); \beta'\rangle \leqslant \langle \mathrm{Pref}_{2n}\,\mathbf{h}(k); \beta'\rangle$, where $k := \arg\min_{j\in\{0,\dots,b^n-1\}}\langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle$. For each $m \in \mathbb{N}$, there exists $k_m \in \mathbb{Z}$ such that $\mathrm{Pref}_{2n+2m}\,\mathbf{h}(k_m) \in \mathrm{Pref}_{2n}\,\mathbf{h}(k)(0\mathcal{A})^m$ by Lemma 4-14. Then

$$\begin{aligned}\inf_{j\in\mathbb{Z}}\langle \mathbf{h}(j); \beta'\rangle \leqslant \inf_{m\in\mathbb{N}}\langle \mathbf{h}(k_m); \beta'\rangle \leqslant \inf_{m\in\mathbb{N}}\big\langle \mathrm{Pref}_n\,\mathbf{h}(k)(00)^m((b-1)0)^\omega; \beta'\big\rangle \\ = \langle \mathrm{Pref}_n\,\mathbf{h}(k); \beta'\rangle. \quad \square\end{aligned}$$

Remark 4-15. We have that

$$\mu_n := \min_{j\in\{0,1,\dots,b^n-1\}}\langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle = \min_{j\in J_{n-1}+b^{n-1}\mathcal{A}}\langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle, \quad (4\text{-}7)$$

where

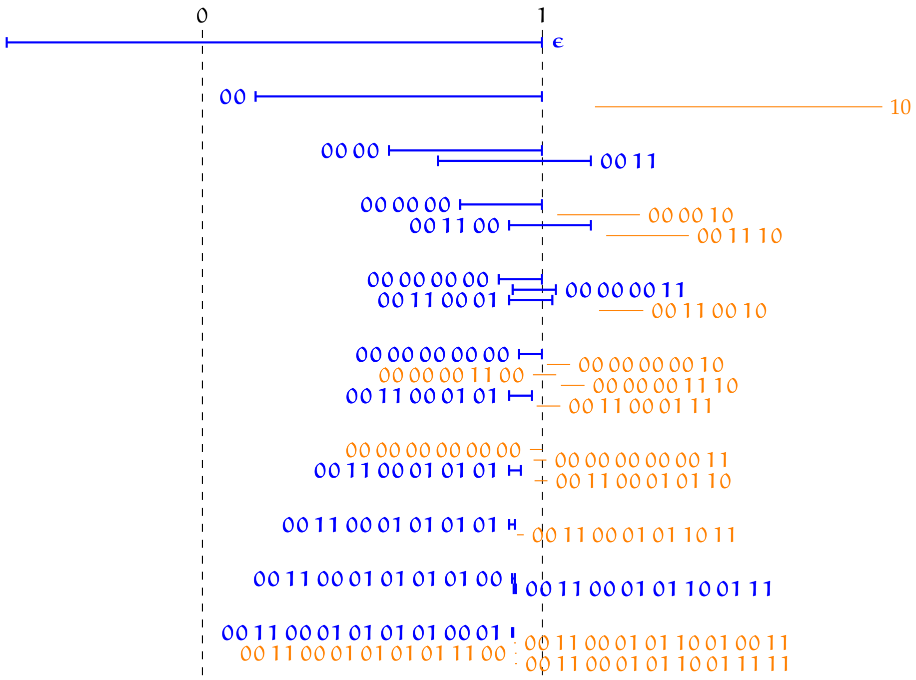
$$\begin{aligned} &J_0 := \{0\},\\ &J_n := \big\{\, j \in J_{n-1} + b^{n-1}\mathcal{A} : \langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle < \mu_n + |\beta'|^{2n+1}\tfrac{b-1}{1-(\beta')^2}\,\big\}.\end{aligned}$$

To verify (4-7), we first show that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is non-increasing. Let $j \in \{0,\dots,b^n-1\}$ be such that $\mu_n = \langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle$. Then by Lemma 4-14 there exists $d \in \mathcal{A}$ such that $\mathrm{Pref}_{2n+1}\,\mathbf{h}(j+db^n) = \mathrm{Pref}_{2n}\,\mathbf{h}(j)0$, whence $\mu_{n+1} \leqslant \langle \mathrm{Pref}_{2n+2}\,\mathbf{h}(j+db^n); \beta'\rangle \leqslant \mu_n$.

Suppose now that $j \in \{0,\dots,b^n-1\} \setminus (J_{n-1} + b^{n-1}\mathcal{A})$. Then there exists $m < n$ such that $\langle \mathrm{Pref}_{2m}\,\mathbf{h}(j); \beta'\rangle \geqslant \mu_m + |\beta'|^{2m+1}\frac{b-1}{1-(\beta')^2}$, therefore $\langle \mathrm{Pref}_{2n}\,\mathbf{h}(j); \beta'\rangle > \mu_m \geqslant \mu_n$.

Example 4-16. As an example, the computation of $\gamma(\beta)$ for $\beta = 1+\sqrt{3}$, the Pisot root of $\beta^2 = 2\beta + 2$, is visualized in Figure 4-4. For each step of the algorithm, the value of $\gamma(\beta)$ lies in the left-most interval. Already in the 5th step we obtain that $\gamma(\beta) \in [0.900834, 0.970552]$, therefore it is strictly between 0 and 1. Note that in the 9th step we have that $\mu_9 = \langle t^{(9)}; \beta'\rangle$ with $t^{(9)} = 001100010101010001$, and $\gamma(\beta) \in [0.91012665225, 0.91587668314]$. In the 40th step, we have that $t^{(40)} = 001100(01)^4000100(0001)^4(00)^2(01)^5(00)^3(01)^6(00)^201$ and $\gamma(\beta) \approx 0.914803044$. In the 200th step, we obtain

$$\gamma(1+\sqrt{3}) \approx 0.914803044196658195047293139393794152694998618733976175733141835762361.$$



(Fig. 4-4) The computation of $\gamma(1+\sqrt{3})$. By a thick line with a bold label we denote the intervals that we 'keep' (these arise from numbers in J_n), by a thin line the ones that we 'forget'. The labels next to the intervals are the corresponding prefixes $\mathrm{Pref}_{2n}\,\mathbf{h}(j)$.

Proof of Theorem 4-5, case $a = b$. Take $a = b \geqslant 4$. Then $b = \beta^2 + (b-1)\beta^3 + (2b+1)\beta^4$, therefore $\mathrm{Pref}_4\,\mathbf{h}(b) = 001(b{-}1)$. According to Proposition 4-13, we have that

$$A := \inf_{j\in\mathbb{Z}} \langle \mathbf{h}(j); \beta' \rangle \leqslant \langle 001(b{-}1); \beta' \rangle = (\beta')^2 + (b-1)(\beta')^3.$$

For $a = b \geqslant 5$, we use the estimate $-\beta' \in (\frac{b}{b+1}, 1)$ to obtain that $A < 1 - \frac{b^3(b-1)}{(b+1)^3} < -1$, therefore $\gamma(\beta) = 0$. For $a = b = 4$, we have $\langle 001(b{-}1); \beta' \rangle \approx -1.0193$, thus $A < -1$.

When $a = b = 3$, we verify that $\mathrm{Pref}_{12}\,\mathbf{h}(21) = 001200020201$ and Proposition 4-13 yields $A \leqslant \langle 001200020201; \beta' \rangle \approx -1.0726 < -1$, therefore $\gamma(\beta) = 0$.

When $a = b = 2$, we can follow the lines of the proof of the case $a > \frac{1+\sqrt{5}}{2}b$, because we observe that (4-5) is satisfied, namely $6 + 8\beta' \approx 0.1436 > 0$. □

The proof of Theorem 4-6 is divided into several cases.

Proof of Theorem 4-6, case $a \geqslant b^2$. Any $j \in \mathbb{Z}\setminus\{0\}$ can be written as $j = b^n(j_0 + j_1 b)$, where $n \in \mathbb{N}$, $j_0 \in \mathcal{A}\setminus\{0\}$ and $j_1 \in \mathbb{Z}$. Then $\mathrm{Pref}_{2n+1}\,\mathbf{h}(j) = 0^{2n}j_0$ because

$b^n \in \beta^{2n} + \beta^{2n+1}\mathbb{Z}[\beta]$, whence

$$\begin{aligned}\langle \mathbf{h}(j);\beta'\rangle &\geqslant \big\langle \mathrm{Pref}_{2n+1}\,\mathbf{h}(j)((b-1)0)^\omega;\beta'\big\rangle \geqslant \big\langle 0^{2n}1((b-1)0)^\omega;\beta'\big\rangle \\ &= (\beta')^{2n}\Big(1+\frac{(b-1)\beta'}{1-(\beta')^2}\Big) = (\beta')^{2n}\Big(1-\frac{(b-1)b\beta}{\beta^2-b^2}\Big) > 0,\end{aligned}$$

where the last inequality was already proved in [**MS14**, Theorem 6]. As $\mathbf{h}(0) = 0^\omega$, we have $\langle \mathbf{h}(0);\beta'\rangle = 0$. From Theorem 4-5 we conclude that $\gamma(\beta) = 1 + \inf_{j\in\mathbb{Z}}\langle \mathbf{h}(j);\beta'\rangle = 1$. □

The remaining cases of the proof of Theorem 4-6 make use of the following relations. Let $c := a/b \in \mathbb{Z}$. Then $\frac{b}{\beta^2} = \frac{1}{1+c\beta} \in 1 - c\beta + c^2\beta^2 - c^3\beta^3 + \beta^4\mathbb{Z}[\beta]$, and more generally,

$$\frac{b^n}{\beta^{2n}} \in 1 - nc\beta + \tbinom{n+1}{2}c^2\beta^2 - \tbinom{n+2}{3}c^3\beta^3 + \beta^4\mathbb{Z}[\beta] \quad \text{for any } n\in\mathbb{N}. \tag{4-8}$$

For $j = (j_0 + j_1 b)b^n$ with $n \in \mathbb{N}$, and $j_0, j_1 \in \mathbb{Z}$ we have that $\frac{j}{\beta^{2n}} = j_0\frac{b^n}{\beta^{2n}} + j_1\beta^2\frac{b^{n+1}}{\beta^{2n+2}}$, therefore

$$\begin{aligned}\frac{j}{\beta^{2n}} \in j_0 - j_0 nc\beta &+ \Big(j_0\tbinom{n+1}{2}c^2 + j_1\Big)\beta^2 \\ &- \Big(j_0\tbinom{n+2}{3}c^3 + j_1(n+1)c\Big)\beta^3 + \beta^4\mathbb{Z}[\beta].\end{aligned} \tag{4-9}$$

Proof of Theorem 4-6, case $\beta^2 = 30\beta + 6$. We have $b = 6$ and $c = 5$. As in the proof of the previous case, we will show that $\langle \mathbf{h}(j);\beta'\rangle \geqslant 0$ for all $j \in \mathbb{Z}$. Let $j \neq 0$ be written as $j = b^n(j_0 + j_1 b)$ with $j_0 \in \mathcal{A}\setminus\{0\}$ and $j_1 \in \mathbb{Z}$, then $\mathbf{h}(j) = 0^{2n}u_0u_1u_2\cdots$ for some $u_0u_1\cdots \in \mathcal{A}^\omega$ with $u_0 = j_0$, and $\langle \mathbf{h}(j);\beta'\rangle = (\beta')^{2n}\langle u_0u_1\cdots;\beta'\rangle$. We consider the following cases:

- If $u_0 \geqslant 2$, then $\langle u_0u_1\cdots;\beta'\rangle \geqslant \langle 2(50)^\omega;\beta'\rangle > 0$.
- If $u_0 = 1$ and $u_1 \leqslant 4$, then $\langle u_0u_1\cdots;\beta'\rangle \geqslant \langle 14(05)^\omega;\beta'\rangle > 0$.
- If $u_0u_1 = 15$, then (4-9) yields that $j_0 = 1$ and $-j_0nc \equiv 5 \pmod 6$, therefore $n \equiv -1 \pmod 6$ and $n = 6n_1 - 1$, i.e., $-j_0nc\beta = 5\beta - 30n_1\beta \in 5\beta - 5n_1\beta^3 + \beta^4\mathbb{Z}[\beta]$. Therefore

$$\begin{aligned}\frac{j}{\beta^{2n}} \in 1 + 5\beta &+ \Big(\tbinom{6n_1}{2}5^2 + j_1\Big)\beta^2 \\ &- \Big(\tfrac{(6n_1+1)6n_1(6n_1-1)}{6}5^3 + 30n_1j_1 + 5n_1\Big)\beta^3 + \beta^4\mathbb{Z}[\beta].\end{aligned}$$

  The coefficient of β^3 is congruent to 0 modulo 6 regardless of the values of n_1 and j_1. This means that $u_3 = 0$. Then $\langle 15u_20(05)^\omega;\beta'\rangle \geqslant \langle 1500(05)^\omega;\beta'\rangle > 0$.

Therefore we have $\langle \mathbf{h}(j);\beta'\rangle \geqslant 0$ for all $j \in \mathbb{Z}$. □

Proof of Theorem 4-6, case $\beta^2 = 24\beta + 6$. We have $b = 6$ and $c = 4$. We use the same technique as in the case $\beta^2 = 30\beta + 6$.

- If $u_0 \geqslant 2$, then $\langle u_0 u_1 \cdots ; \beta' \rangle \geqslant \langle 2(50)^\omega ; \beta' \rangle > 0$.
- If $u_0 = 1$ and $u_1 \leqslant 3$, then $\langle u_0 u_1 \cdots ; \beta' \rangle \geqslant \langle 13(05)^\omega ; \beta' \rangle > 0$.
- Since c is even, we get that $u_1 \equiv -j_0 nc \pmod 6$ is even, therefore $u_0 u_1 \neq 15$.
- If $u_0 u_1 = 14$, then (4-9) gives $j_0 = 1$ and $-j_0 nc \equiv 4 \pmod 6$, i.e., $n \equiv -1 \pmod 3$ and $n = 3n_1 - 1$, whence $-j_0 nc\beta = 4\beta - 12 n_1 \beta \in 4\beta - 2n_1\beta^3 + \beta^4\mathbb{Z}[\beta]$. We derive that

  $$\frac{j}{\beta^{2n}} \in 1 + 4\beta + (\text{some integer})\beta^2 - \big(144 n_1^3 - 30 n_1 + 12 n_1 j_1\big)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

  As above, we get that $u_3 = 0$ regardless of the values of n_1 and j_1, thus $\langle u_0 u_1 \cdots ; \beta' \rangle \geqslant \langle 1400(05)^\omega ; \beta' \rangle > 0$. $\square$

Proof of Theorem 4-6, case $c := a/b < b$ *and* $c \notin \{4,5\}$ *when* $b = 6$. Let $n := \lceil \frac{c}{b-c} \rceil$. From (4-8), the β-adic expansion $\mathbf{h}(b^n)$ starts with $0^{2n}1(nb{-}nc)$. If $\frac{c}{b-c} \notin \mathbb{Z}$, then we have $nb - nc > c$ and thus $\langle 1(nb{-}nc); \beta' \rangle \leqslant 1 + (c+1)\beta' < 0$, using that $\beta' = -\frac{b}{\beta} < -\frac{b}{cb+1} \leqslant -\frac{1}{c+1}$. By Proposition 4-13, this proves that $\gamma(\beta) < 1$ if c is not a multiple of $b - c$.

Assume now that $\frac{c}{b-c} \in \mathbb{Z}$, i.e., $n = \frac{c}{b-c}$. For $j := b^n - \binom{n+1}{2} c^2 b^{n+1}$, we have by (4-9) that

$$\frac{j}{\beta^{2n}} \in 1 - nc\beta - \left(\binom{n+2}{3}c^3 - \binom{n+1}{2}c^3(n+1)\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

Since $-nc = c - nb \in c - n\beta^2 + \beta^3\mathbb{Z}[\beta]$ and $(n+1)c = nb \in \beta\mathbb{Z}[\beta]$, we obtain that

$$\frac{j}{\beta^{2n}} \in 1 + c\beta - \left(\binom{n+2}{3}c^3 + n\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

If $\binom{n+2}{3}c^3 + n \not\equiv 0 \pmod b$, then

$$\begin{aligned}\langle \mathrm{Pref}_{2n+4}\,\mathbf{h}(j); \beta' \rangle \leqslant \langle 0^{2n}1c01; \beta' \rangle &= (\beta')^{2n}\big(1 + c\beta' + (\beta')^3\big) \\ &= \frac{(\beta')^{2n+2}}{b} + (\beta')^{2n+3} = (\beta')^{2n+2}\frac{\beta - b^2}{b\beta} < 0,\end{aligned}$$

since $1 + c\beta' = \frac{(\beta')^2}{b}$ and $\beta < a + 1 \leqslant b^2$, therefore $\gamma(\beta) < 1$ by Proposition 4-13.

It remains to consider the case that $\binom{n+2}{3}c^3 + n \equiv 0 \bmod b$, i.e.,

$$n \equiv -\frac{bn(n+2)}{6}c^2 n \bmod b,$$

because $(n+1)c = nb$. Multiplying by $b - c$ gives

$$c \equiv -\frac{bn(n+2)}{6}c^3 \bmod b.$$

Note that $\frac{bn(n+2)}{6} = (b-c)\binom{n+2}{3} \in \mathbb{Z}$. We distinguish four cases:

1. If $6 \perp b$, then $c \equiv 0 \bmod b$, contradicting that $1 \leqslant c < b$.
2. If $2 \mid b$ and $3 \nmid b$, then c is a multiple of $b/2$, i.e., $c = b/2$, $n = 1$. As n is also a multiple of $b/2$, we get that $b = 2$, thus $c = 1$. For $\beta^2 = 2\beta + 2$, we already know that $\gamma(\beta) < 1$, see Example 4-16.
3. If $3 \mid b$ and $2 \nmid b$, then c and n are multiples of $b/3$. For $c = b/3$ we have $n \notin \mathbb{Z}$. For $c = 2b/3$, we have $n = 2$, thus $b \in \{3, 6\}$. However, $b = 6$ contradicts $2 \nmid b$ and $b = 3$ (i.e., $c = 2$) contradicts $\binom{n+2}{3}c^3 + n \equiv 0 \bmod b$.
4. If $6 \mid b$, then c and n are multiples of $b/6$, thus $c \in \{b/2, 2b/3, 5b/6\}$, $n \in \{1, 2, 5\}$. If $n = 1$, then $b = 6$, thus $c = 3$, and $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. If $n = 2$, then $b \in \{6, 12\}$; we have excluded that $b = 6$, $c = 4$; for $b = 12$, $c = 8$, we have $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. If $n = 5$, then $b \in \{6, 30\}$; we have excluded that $b = 6$, $c = 5$; for $b = 30$, $c = 24$, we have $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. □

4-7 The general quadratic case

In the general quadratic case where $2 \leqslant \gcd(a, b) \leqslant b - 1$, the conditions of Theorem 4-5 need not be satisfied. This means that we have to rely on the more general Theorem 4-4, i.e., to compute the two values $\inf_{j\in\mathbb{Z}}\langle \mathbf{h}(j); \beta'\rangle$ and $\sup_{j\in\mathbb{Z}}\langle \mathbf{h}(j - \beta); \beta'\rangle$.

We can derive, in a similar manner to Proposition 4-11, that for all $n \in \mathbb{N}$,

$$\sup_{j\in\mathbb{Z}}\langle \mathbf{h}(j - \beta); \beta'\rangle \in \max_{j\in\{0,1,\ldots,b^n-1\}}\langle \mathrm{Pref}_n\, \mathbf{h}(j - \beta); \beta'\rangle + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]. \quad (4\text{-}10)$$

Let now $s_n \geqslant 1$, for $n \in \mathbb{N}$, denote the smallest positive integer such that $s_n \in \beta^n\mathbb{Z}[\beta]$, and $r_n := \frac{s_n}{s_{n-1}}$. Then $x, y \in \mathbb{Z}$ have a common prefix of length n if and only if $y - x \in s_n\mathbb{Z}$. Therefore, in both (4-6) and (4-10) we can take $\{0, 1, \ldots, s_n - 1\}$ instead of $\{0, 1, \ldots, b^n - 1\}$. Moreover, following Remark 4-15, we can further restrict to the sets

$$\begin{aligned} J_0 &:= \{0\}, \qquad J_0' := \{-\beta\},\\ J_n &:= \big\{\, j \in J_{n-1} + s_{n-1}\{0, \ldots, r_n - 1\} : \langle \mathrm{Pref}_n\, \mathbf{h}(j); \beta'\rangle \leqslant \mu_n + |\beta'|^n \tfrac{b-1}{1+\beta'}\,\big\},\\ J_n' &:= \big\{\, j \in J_{n-1} + s_{n-1}\{0, \ldots, r_n - 1\} : \langle \mathrm{Pref}_n\, \mathbf{h}(j); \beta'\rangle \geqslant \nu_n - |\beta'|^n \tfrac{b-1}{1+\beta'}\,\big\}, \end{aligned}$$

where

$$\mu_n := \min_{j\in\{0,\ldots,b^n-1\}}\langle \mathrm{Pref}_n\, \mathbf{h}(j); \beta'\rangle, \quad \nu_n := \max_{j\in\{0,\ldots,b^n-1\}}\langle \mathrm{Pref}_n\, \mathbf{h}(j - \beta); \beta'\rangle.$$

4-8 Continuation of the work

We conclude by several open questions and problems that arise in the study of rational numbers with purely periodic expansions:

Problem 4a. Prove or disprove that $\gamma(\beta) = 1$ for quadratic Pisot number $\beta > 1$, a root of $\beta^2 = a\beta + b$, if and only if $\frac{a}{b} \in \mathbb{Z}$ and $a \geqslant b^2$ or $(a, b) \in \{(24, 6), (30, 6)\}$.

Problem 4b. For which quadratic β we have that $\gamma(\beta) = 0$? Can we drop the restrictions on a and b in Theorem 4-5? More specifically, is it true that $a < \frac{1+\sqrt{5}}{2} b$ implies $\gamma(\beta) = 0$?

Problem 4c. What is the structure of the prefixes of β-adic expansions of integers for a general quadratic β?

Problem 4d. What about the cubic Pisot case? S. Akiyama and K. Scheicher [**AS05**] showed how to compute the value $\gamma(\beta)$ for $\beta = \varphi_p$ the minimal Pisot number. B. Loridant et al. [**LMST13**] gave the contact graph of the β-tiles for cubic units, which could be used to determine $\gamma(\beta)$ for the units, in a similar way to what Akiyama and Scheicher did. The consideration of the β-adic spaces could then allow the results to be expanded to non-units as well.

CHAPTER FIVE

Spectra of complex numbers

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5-1 Introduction

A *spectrum* of a number β that is > 1 in modulus with a fixed finite alphabet $\mathcal{A} \subset \mathbb{C}$ is the set of all linear combinations of powers of β with coefficients in the alphabet:

$$\mathcal{A}[\beta] := \{\, x_0 + x_{-1}\beta + x_{-2}\beta^2 + \cdots + x_{-N}\beta^N : N \in \mathbb{N},\ x_i \in \mathcal{A} \,\}.$$

As many authors before, we restrict in most results the alphabet to the form $\mathcal{A}_m := \{0, 1, \ldots, m\}$. The biggest interest is in exploring the Delone properties of spectra, i.e., in knowing under what conditions the spectrum is uniformly discrete and relatively dense, and in determining the minimal and maximal distances between consecutive points. In the real case $\beta > 1$, $\mathcal{A}_m[\beta] \subset \mathbb{R}_+$ has no accumulation points, therefore we can enumerate its elements — there exists an increasing sequence

$$0 =: x_0 < x_1 < x_2 < \cdots \tag{5-1}$$

such that $\mathcal{A}_m[\beta] = \{\, x_k : k \in \mathbb{N} \,\}$. Several authors have been interested in determining the values $\ell_m(\beta) := \liminf(x_{k+1} - x_k)$ and $L_m(\beta) := \limsup(x_{k+1} - x_k)$. Values $\ell_1(\beta)$ and $L_1(\beta)$ have been studied by P. Erdős et al. [**EJK90**, **EJJ92**, **EJK98**]. The value of $\ell_m(\beta)$ for all m was first determined for $\beta = \varphi_g$ the Golden mean by V. Komornik, P. Loreti and M. Pedicini [**KLP00**] and then for all quadratic Pisot units by T. Komatsu [**Kom02**]; $L_m(\varphi_g)$ was determined for all m at once by P. Borwein and K. Hare [**BH03**]. Z. Masáková, K. Pastirčáková and E. Pelantová [**MPP15**] show, for all quadratic Pisot units β and all $m \geqslant \lfloor \beta \rfloor$, which distances between consecutive points of the spectra appear infinitely many times. Another approach was taken by Borwein and Hare, and D.-J. Feng and Z.-Y. Wen [**BH02**, **FW02**] who independently provided an algorithm that can be used to determine the values $\ell_m(\beta)$ and $L_m(\beta)$ for a fixed m and β.

In this chapter, we concentrate on the spectra of complex numbers. Since in the complex plane, we cannot enumerate the elements of the spectra as nicely as in (5-1), we have to figure out a different approach to defining $\ell_m(\gamma)$ and $L_m(\gamma)$ for a complex number γ that is > 1 in modulus.

Definition 5-1. Let $\gamma \in \mathbb{C} \setminus \mathbb{R}$ and $m \in \mathbb{N}$, $m \geqslant 1$. We denote

$$\ell_m(\gamma) := \inf\{\, |x - y| : x, y \in \mathcal{A}_m[\gamma],\ x \neq y \,\};$$
$$L_m(\gamma) := \sup\{\, D \geqslant 0 : \exists z \in \mathbb{C} \text{ such that } B_D(z) \cap \mathcal{A}_m[\gamma] = \emptyset \,\}.$$

We immediately see that $\ell_m(\gamma) > 0$ if and only if $\mathcal{A}_m[\gamma]$ is uniformly discrete and $L_m(\gamma) < \infty$ if and only if it is relatively dense (cf. § 2-4).

5-2 Main results

In this chapter, we present two main results. The first is very general and applies to all complex spectra arbitrary alphabets (certainly it applies to the alphabets $\mathcal{A}_m$):

Theorem 5-2. *Let $\gamma \in \mathbb{C}$ be a non-real number > 1 in modulus. Suppose $\mathcal{A} \subseteq \mathbb{C}$ is an alphabet with $0 \in \mathcal{A}$ whose cardinality satisfies $\#\mathcal{A} < |\gamma|^2$. Then the spectrum $\mathcal{A}[\gamma]$ is not relatively dense.*

This is a complex counterpart to a result of P. Erdős and V. Komornik [**EK98**] showing that the same is true for β real and $\#\mathcal{A} < \beta$ (they show it for integer alphabets, the generalization to arbitrary alphabets is straightforward).

We also provide an algorithm for obtaining the value of $\ell_m(\gamma)$ and $L_m(\gamma)$ for all m at once, given a cubic complex Pisot unit γ with the following specific property:

Definition 5-3. We say that a cubic complex Pisot unit γ satisfies *Property* (F′) if the number $1/\gamma'$ is positive and satisfies Property (F) for greedy β-expansions. Here, we denote $\gamma' \in \mathbb{R}$ the unique real Galois conjugate of γ.

The algorithm is described in detail in Algorithms 5-12 and 5-17. Applied to the case of the complex Tribonacci constant, we get the following:

Theorem 5-4. *Let $\gamma = \gamma_t \approx -0.771 + 1.115i$ be a root of $X^3 + X^2 + X - 1 = 0$, let $m \in \mathbb{N} \setminus \{0\}$, and let $k \in \mathbb{Z}$ be the greatest integer such that $m \geqslant (1 - \gamma_t')\big(\frac{1}{\gamma_t'}\big)^k$, where γ_t' is the real Galois conjugate of γ_t. Then we have*

$$\ell_m(\gamma_t) = |\gamma_t|^{-k} \quad \textit{and} \quad L_m(\gamma_t) = 2\sqrt{\frac{1 - (\gamma_t')^2}{3 - (\gamma_t')^2}}\,|\gamma_t|^{3-k}. \tag{5-2}$$

The set of cubic units satisfying Property (F) for greedy β-numeration was described by S. Akiyama [**Aki00**, Theorem 3]. From his result, we deduce that γ root of $X^3 = -bX^2 - aX + 1$ satisfies Property (F′) if and only if

$$|b - 1| \leqslant a, \quad b \geqslant -1 \quad \text{and} \quad -18ab - 4a^3 + a^2b^2 + 4b^3 - 27 < 0.$$

The first two conditions are due to Akiyama, the last one is a condition on the polynomial determinant that assures the polynomial has complex roots. This implies that there are infinitely many numbers γ satisfying (F′); for instance, all cases $a \geqslant 1$ and $b = 0, \pm 1$, with the exception $(a, b) = (1, -1)$.

5-3 Proof of Theorem 5-2

To prove Theorem 5-2, we cannot easily follow the lines of the proof of the result for the real case (i.e., that $\#\mathcal{A} < \beta$ implies $L_m(\beta) = +\infty$), because it relies on the natural ordering of $\mathbb{R}$. So we have to use a different technique, based on the following 'folklore' lemma about the asymptotic density of relatively dense sets:

Lemma 5-5. *Let $\Lambda \subset \mathbb{C}$ be a relatively dense set. Then*

$$\liminf_{r\to\infty} \frac{\#(\Lambda \cap B_r)}{r^2} > 0, \tag{5-3}$$

where $B_r := \{ z \in \mathbb{C} : |z| < r \}$ is the ball of radius r centered at 0.

Proof. Since Λ is relatively dense, there exists $\lambda > 0$ such that every square in $\mathbb{C}$ with side λ contains a point of Λ. Therefore every cell of the lattice $\lambda\mathbb{Z}[i] = \{ \lambda a + i\lambda b : a, b \in \mathbb{Z}\}$ contains a point of Λ. Since B_r contains at least n^2 cells, where $n = \lfloor r\sqrt{2}/\lambda \rfloor$, we get

$$\liminf_{r\to\infty} \frac{\#(\Lambda \cap B_r)}{r^2} \geqslant \liminf_{r\to\infty} \frac{\lfloor r\sqrt{2}/\lambda \rfloor^2}{r^2} = \frac{2}{\lambda^2} > 0. \qquad \square$$

Proof of Theorem 5-2. For simplicity, we denote $\Lambda := \mathcal{A}[\gamma]$ the spectrum, and $c := \max_{a\in\mathcal{A}}|a|$.

First, we show that for any $r \geqslant c$ we have

$$\Lambda \cap B_{|\gamma|r-c} \subseteq \gamma(\Lambda \cap B_r) + \mathcal{A}$$

and therefore

$$\#(\Lambda \cap B_{|\gamma|r-c}) \leqslant \#\mathcal{A}\,\#(\Lambda \cap B_r). \tag{5-4}$$

To prove this, consider $x = \sum_{j=0}^{k} a_j\gamma^j$ with $a_j \in \mathcal{A}$ and such that $|x| < |\gamma|r - c$. Then $y := (x - a_0)/\gamma = \sum_{j=1}^{k} a_j\gamma^{j-1} \in \Lambda$ and $|y| \leqslant (|x| + a_0)/|\gamma| < (|\gamma|r - c + c)/|\gamma| = r$. Since $x = \gamma y + a_0$, the inclusion is valid.

Our aim is to prove that under the assumption $\#\mathcal{A} < |\gamma|^2$, the set Λ is not relatively dense. According to Lemma 5-5, it is enough to construct a sequence (r_k) such that $r_k \to \infty$ and

$$\lim_{k\to\infty} \frac{\#(\Lambda \cap B_{r_k})}{r_k^2} = 0.$$

Consider a sequence given by the recurrence relation $r_{k+1} = |\gamma|r_k - c$ and $r_0 := 1 + \frac{c}{|\gamma|-1} > c$. The choice of r_0 guarantees that $r_k = |\gamma|^k + \frac{c}{|\gamma|-1}$, therefore

$r_k \to \infty$ and $r_{k+1}/r_k \to |\gamma|$. Eventually $r_k \geqslant c$ and (5-4) gives $\#(\Lambda \cap B_{r_{k+1}})) \leqslant \#\mathcal{A}\,\#(\Lambda \cap B_{r_k}))$, which yields

$$\frac{\#(\Lambda \cap B_{r_{k+1}})/r_{k+1}^2}{\#(\Lambda \cap B_{r_k})/r_k^2} \leqslant \frac{\#\mathcal{A}\, r_k^2}{r_{k+1}^2} \xrightarrow{k\to\infty} \frac{\#\mathcal{A}}{|\gamma|^2} < 1,$$

therefore $\#(\Lambda \cap B_{r_k})/r_k^2 \to 0$ as desired. □

5-4 Model sets versus $\mathcal{A}_m[\gamma]$

We recall from (2-1) that to a real Pisot number β of degree three such that β has one pair of complex Galois conjugates β' and $(\beta')^\dagger$, the associated model (cut-and-project) set is

$$\Lambda_\beta(\Omega) = \{ z \in \mathbb{Z}[1/\gamma] : z' \in \Omega \}, \quad \text{where } \Omega \subseteq \mathbb{R} \text{ is an interval} \tag{5-5}$$

and where we denote $\gamma = 1/\beta'$. We now show how $\mathcal{A}_m[\gamma]$ fits into the cut-and-project scheme:

Theorem 5-6. *Let γ be a cubic complex Pisot unit with Property (F′), i.e., with a real positive conjugate γ' such that $1/\gamma'$ has Property (F). Let m be an integer $m \geqslant |\gamma|^2 - 1$. Then $\mathcal{A}_m[\gamma]$ is a model set, namely*

$$\mathcal{A}_m[\gamma] = \Lambda_\beta(\Omega) = \{ z \in \mathbb{Z}[\gamma] : z' \in \Omega \} \quad \textit{with } \Omega = [0, m/(1-\gamma')). \tag{5-6}$$

Proof. Inclusion $\subseteq$: Let $z \in \mathcal{A}_m[\gamma]$. Then $z = \sum_{j=0}^n a_j \gamma^j$ with $a_j \in \{0, \ldots, m\}$ and clearly $z \in \mathbb{Z}[\gamma]$. Moreover,

$$0 \leqslant z' = \sum_{j=0}^{n} a_j (\gamma')^j \leqslant \sum_{j=0}^{n} m(\gamma')^j < \frac{m}{1-\gamma'}.$$

Inclusion $\supseteq$: Let us take $z \in \mathbb{Z}[\gamma]$ with $z' \in \Omega$. Denote $\beta = 1/\gamma' = \gamma\gamma^\dagger = |\gamma|^2$. We discuss the following two cases:

1. Suppose $0 \leqslant z' < 1$. The real base β has Property (F) by the hypothesis. Therefore every number from $\mathbb{Z}[1/\beta] \cap [0,1)$ has a finite expansion $\bullet a_1 a_2 a_3 \ldots a_n$ over the alphabet $\{0, \ldots, m_0\}$, where $m_0 := \lfloor \beta \rfloor$. This means that $z' = \sum_{j=1}^n a_j \beta^{-j}$ and therefore $z = \sum_{j=1}^n a_j \gamma^j \in \mathcal{A}_{m_0}[\gamma]$. Since $\mathcal{A}_{m_0}[\gamma] \subseteq \mathcal{A}_m[\gamma]$, we get $z \in \mathcal{A}_m[\gamma]$.
2. Suppose $1 \leqslant z' < m/(1-\gamma')$. Since $z' < \sum_{j=0}^\infty m\beta^{-j}$, there exists a minimal $k \geqslant 0$ such that $z' - \sum_{j=0}^k m\beta^{-j} < 0$. Let $b \in \{0, \ldots, m\}$ be such that

$$0 \leqslant z' - \sum_{j=0}^{k-1} m\beta^{-j} - b\beta^{-k} < \beta^{-k},$$

where $\sum_{j=0}^{-1} m\beta^{-j} := 0$. Then

$$u' := \beta^k \Big(z' - \sum_{j=0}^{k-1} m\beta^{-j} - b\beta^{-k} \Big)$$

satisfies $0 \leqslant u' < 1$, and by the previous case there exist $a_1, \dots, a_n \in \{0, \dots, m_0\}$ such that $u' = \sum_{j=1}^{n} a_j \beta^{-j}$. Altogether,

$$z' = \sum_{j=0}^{k-1} m(\gamma')^j + b(\gamma')^k + \sum_{j=k+1}^{k+n} a_{j-k}(\gamma')^j$$

and $z \in \mathcal{A}_m[\gamma]$. □

The property of cut-and-project sets which allows us to determine the values of $\ell_m(\gamma)$ and $L_m(\gamma)$ is the self-similarity. We say that a Delone set $\Lambda \subseteq \mathbb{C}$ is *self-similar* with a factor $\kappa \in \mathbb{C}$, $|\kappa| > 1$, if $\kappa\Lambda \subseteq \Lambda$. In general, cut-and-project sets are not self-similar. In our special case (5-5), not only the sets are self-similar, but we can prove even a stronger property that will be useful later:

Proposition 5-7. *Let γ be a cubic complex Pisot unit. Then*

$$\Lambda\big((\gamma')^k\Omega\big) = \gamma^k \Lambda(\Omega) \quad \textit{for any interval } \Omega \textit{ and any } k \in \mathbb{Z}.$$

In particular, if $\Omega = [0, c)$ and γ' is positive, then $\gamma'\Omega \subseteq \Omega$ and $\gamma\Lambda \subseteq \Lambda$.

Proof. We prove the claim for $k = \pm 1$, the general case follows by induction. Because $\mathbb{Z}[\gamma] = \gamma\mathbb{Z}[\gamma]$, we have that

$$\begin{aligned}\Lambda(\gamma'\Omega) &= \big\{ x \in \gamma\mathbb{Z}[\gamma] : x' \in \gamma'\Omega \big\} = \big\{ x \in \gamma\mathbb{Z}[\gamma] : \tfrac{1}{\gamma'}x' \in \Omega \big\} \\ &= \gamma\big\{ y \in \mathbb{Z}[\gamma] : y' \in \Omega \big\} = \gamma\Lambda(\Omega), \end{aligned} \tag{5-7}$$

which implies the validity of the statement for $k = +1$. If we apply (5-7) to the window $\tilde{\Omega} = \gamma'\Omega$, we get $\Lambda(\tilde{\Omega}) = \gamma\Lambda(\frac{1}{\gamma'}\tilde{\Omega})$, i.e., $\frac{1}{\gamma}\Lambda(\tilde{\Omega}) = \Lambda(\frac{1}{\gamma'}\tilde{\Omega})$, which implies the validity of the statement for $k = -1$. □

5-5 Voronoi tiling of model sets

We recall that for a Delone set $\Lambda \subseteq \mathbb{C}$, the Voronoi tile of a point $x \in \Lambda$ is the set of points which are closer to x than to any other point in Λ. Formally

$$\mathcal{V}(x) = \big\{ z \in \mathbb{C} : |z - x| \leqslant |z - y| \text{ for all } y \in \Lambda \big\}. \tag{5-8}$$

The tile is a convex polygon having x as an interior point, and $\{\mathcal{V}(x)\}_{x \in \Lambda}$ is a tiling of $\mathbb{C}$. For every tile $\mathcal{V}(x)$ we define two characteristics:

- $\delta(\mathcal{V}(x))$ is the maximal diameter $d > 0$ such that $B_{d/2}(x) \subseteq \mathcal{V}(x)$;
- $\Delta(\mathcal{V}(x))$ is the minimal diameter $D > 0$ such that $\mathcal{V}(x) \subseteq B_{D/2}(x)$.

These δ and Δ allow us to compute the values of $\ell_m(\gamma)$ and $L_m(\gamma)$, namely

$$\ell_m(\gamma) = \inf_x \delta\big(\mathcal{V}(x)\big) \quad \text{and} \quad L_m(\gamma) = \sup_x \Delta\big(\mathcal{V}(x)\big),$$

where x runs the whole set $\Lambda = \mathcal{A}_m(\gamma)$.

A *prototile* of a point x is the set $\mathcal{V}(x) - x$. We can define δ, Δ analogously for the prototiles. The set of all prototiles of the tiling of Λ is called the *palette* of Λ. We therefore obtain that

$$\ell_m(\gamma) = \inf_{\mathcal{V}} \delta(\mathcal{V}) \quad \text{and} \quad L_m(\gamma) = \sup_{\mathcal{V}} \Delta(\mathcal{V}), \tag{5-9}$$

where $\mathcal{V}$ runs the whole palette of Λ.

For computing $\delta(\mathcal{V})$ and $\Delta(\mathcal{V})$, we modify the approach of [**MPZ03**], where 2-dimensional cut-and-project sets based on quadratic irrationalities are concerned. To find the Voronoi tile of a point $x \in \Lambda(\Omega)$ one does not need to consider all points $y \in \Lambda(\Omega)$. It is easy to see that only points y closer to x than $\Delta(\mathcal{V}(x))$ influence the shape of the tile $\mathcal{V}(x)$, i.e.,

$$\mathcal{V}(x) = \big\{\, z \in \mathbb{C} : |z - x| \leqslant |z - y| \text{ for } y \in \Lambda(\Omega), |y - x| \leqslant \Delta(\mathcal{V}(x)) \,\big\}. \tag{5-10}$$

But before the shape of $\mathcal{V}(x)$ is known, we do not know the value of $\Delta(\mathcal{V}(x))$. So we need to find some positive constant L such that

$$\Delta\big(\mathcal{V}(y)\big) \leqslant L \quad \text{for all} \quad y \in \Lambda(\Omega). \tag{5-11}$$

In the rest of this section, we consider cut-and-project sets $\Lambda(\Omega)$ as given by (5-5), where γ has Property (F′), i.e., $1/\gamma'$ has Property (F), and where $\Omega = [0, c)$ with $c > 0$ (however, not necessarily of the form $c = \frac{m}{1-\gamma'}$). We denote by $\operatorname{Re} z = \frac{z+z^\dagger}{2}$ and $\operatorname{Im} z = \frac{z-z^\dagger}{2i}$ respectively the real and the imaginary part of $z \in \mathbb{C}$.

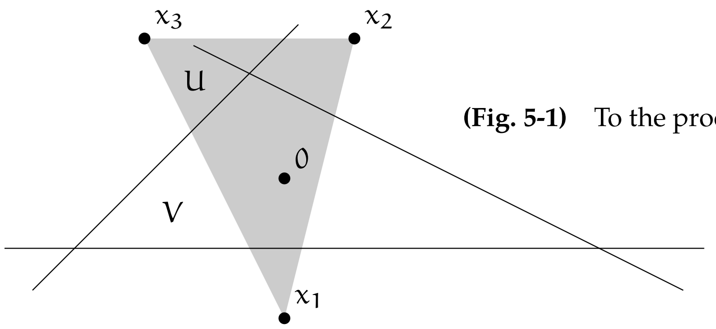
Lemma 5-8. *Let $\Omega = [0, c)$ be an interval. Let p be the first positive integer such that $\operatorname{Im}(\gamma^p)$ and $\operatorname{Im}\gamma$ have the opposite signs and let k be the smallest integer satisfying $(\gamma')^k < c/2$. Then*

$$L := |\gamma|^k \max_{\substack{i,j \in \{0, p-1, p\} \\ i<j}} \left| \frac{\gamma^{i+j}(\gamma^i - \gamma^j)}{\operatorname{Im}((\gamma^j)^\dagger \gamma^i)} \right| \tag{5-12}$$

satisfies $\Delta(\mathcal{V}(y)) \leqslant L$ for all $y \in \Lambda(\Omega)$.

Proof. We first prove the statement for $y = 0$. The choice of k guarantees that $x_1 := \gamma^k$, $x_2 := \gamma^{k+p-1}$ and $x_3 := \gamma^{k+p}$ satisfy $x_1, x_2, x_3 \in \Lambda(\Omega)$, whereas the choice of p guarantees that 0 is an inner point of the triangle U with vertices x_1, x_2, x_3 (see Figure 5-1). According to (5-8) we have

$$V := \big\{\, z \in \mathbb{C} : |z - 0| \leqslant |z - x_j| \text{ for } j = 1, 2, 3 \,\big\} \supseteq \mathcal{V}(0).$$



(Fig. 5-1) To the proof of Lemma 5-8.

Let ρ be the radius of the smallest ball centered at 0 and containing the whole triangle V. From the definition of $\mathcal{V}(x)$ and $\Delta(\mathcal{V}(x))$ we see that $\Delta(\mathcal{V}(0)) \leqslant 2\rho$.

The vertices of V are the points v_{12}, v_{23}, v_{31} such that

$$|x_i - v_{ij}| = |x_j - v_{ij}| = |0 - v_{ij}|. \tag{5-13}$$

These equations have a unique solution

$$v_{ij} = i\frac{x_i x_j (x_i^\dagger - x_j^\dagger)}{2\,\mathrm{Im}(x_i x_j^\dagger)}, \quad \text{whence} \quad |v_{ij}| = \frac{1}{2}\left|\frac{x_i x_j (x_i - x_j)}{\mathrm{Im}(x_i x_j^\dagger)}\right|. \tag{5-14}$$

Then $\rho = \max|v_{ij}|$, thus the estimate (5-12) is valid for $y = 0$ and it remains to show that it is valid for all $y \in \Lambda(\Omega)$. If $y' \in [0, c/2)$ then the three points $y + x_j$ for $j = 1,2,3$ are in $\Lambda(\Omega)$. If $y' \in [c/2, c)$ then the three points $y - x_j$ for $j = 1,2,3$ are in $\Lambda(\Omega)$. Both of these cases follow from the fact that $x_1', x_2', x_3' \in (0, c/2)$. Therefore either x_1, x_2, x_3 or $-x_1, -x_2, -x_3$ are elements of $\Lambda(\Omega) - y$, which means that the same estimate (5-12) can be used. □

To describe the palette of $\Lambda(\Omega)$, we find all possible L-patches, i.e., the local configurations around the points of $\Lambda(\Omega)$ up to a distance L. We recall that an L-patch of $x \in \Lambda(\Omega)$ is the set

$$\Pi_L(x) := \big(\Lambda(\Omega) \cap B_L(x)\big) - x. \tag{5-15}$$

Since we consider the window $\Omega = [0, c)$, the L-patch equals

$$\Pi_L(x) = \{\, z \in \mathbb{Z}[\gamma] : x' + z' \in [0, c) \text{ and } |z| \leqslant L \,\}. \tag{5-16}$$

Lemma 5-9. *Let* $x, y \in \Lambda(\Omega)$ *with* $\Omega = [0, c)$ *and* L *satisfying* (5-11). *Then the equality of two* L*-patches* $\Pi_L(x) = \Pi_L(y)$ *implies the equality of the prototiles, i.e.,* $\mathcal{V}(x) - x = \mathcal{V}(y) - y$.

Proof. Using (5-10) we can write

$$\mathcal{V}(x) = \big\{\, z \in \mathbb{C} : |z - x| \leqslant |z - v| \text{ for all } v \in \Lambda(\Omega) \cap B_L(x) \,\big\}$$

and thus

$$\mathcal{V}(x) - x = \big\{\, s \in \mathbb{C} : |s| \leqslant |s - w| \text{ for all } w \in \Pi_L(x) \,\big\},$$

which depends only on $\Pi_L(x)$ and not on x itself. □

Lemma 5-10. *Let* $x, y \in \Lambda(\Omega)$ *with* $\Omega = [0, c)$ *and* $L > 0$*. If* $\Pi_L(x) \neq \Pi_L(y)$ *then there exists* ξ *from the following finite subset of* $[0, c]$*:*

$$\Xi := \{\, z' : z \in \Pi_L(0) \,\} \cup \{\, c - z' : z \in \Pi_L(0) \,\}, \tag{5-17}$$

such that ξ *lies between* x' *and* y'*, more precisely,* $\min\{x', y'\} < \xi \leqslant \max\{x', y'\}$.

Proof. Without loss of generality, suppose that there exists z such that $z \in \Pi_L(x)$ and $z \notin \Pi_L(y)$. According to (5-16) we have $|z| \leqslant L$, $x' + z' \in [0, c)$, and $y' + z' \notin [0, c)$.

If $x' < y'$ then $x' + z' < c \leqslant y' + z'$, therefore $0 \leqslant x' < c - z' \leqslant y' < c$ and thus x' and y' are separated by $\xi := c - z'$. We have that $c - z' \in (0, c)$, or equivalently $z' \in (0, c)$. As $|z| \leqslant L$, we conclude that $z \in \Pi_L(0)$.

If $x' > y'$ then $y' + z' < 0 \leqslant x' + z'$, therefore $0 \leqslant y' < -z' \leqslant x' < c$ and thus x' and y' are separated by $\xi := -z'$. We have that $-z' \in (0, c)$. As $|-z| = |z| \leqslant L$, we conclude that $-z \in \Pi_L(0)$. □

The two lemmas enable us to partition the interval Ω into sub-intervals such that the points of $\Lambda(\Omega)$ whose Galois conjugates lie in the same sub-interval have the same prototile, formally:

Corollary 5-11. *Let* $\Omega = [0, c)$ *be an interval. Then there exists a finite set* $\Xi = \{\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = c\}$ *such that the mapping*

$$x' \mapsto \mathcal{V}(x) - x$$

is constant on $[\xi_{j-1}, \xi_j) \cap \mathbb{Z}[\gamma']$ *for each* $j = 1, \ldots, N$.

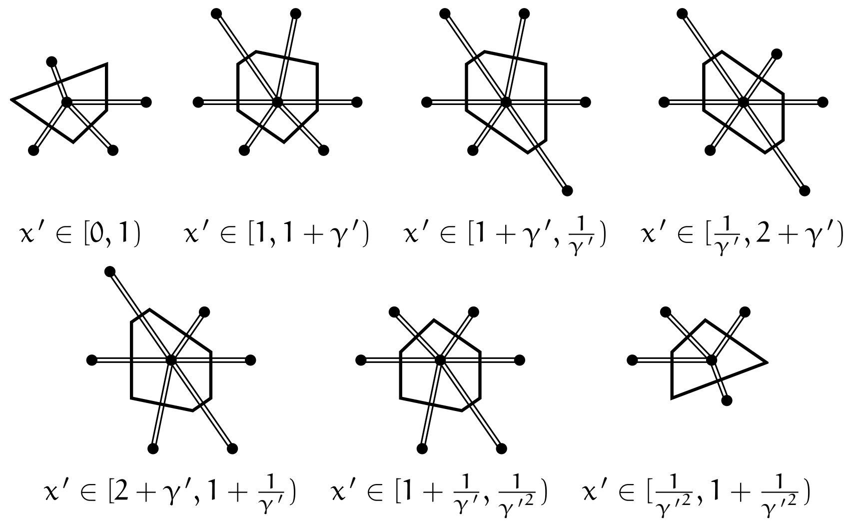
Proof. Consider L satisfying (5-11) and let Ξ be given by (5-17). Suppose $x, y \in \Lambda(\Omega)$ satisfy $x', y' \in [\xi_{j-1}, \xi_j)$. According to Lemma 5-10 we have $\Pi_L(x) = \Pi_L(y)$. Therefore by Lemma 5-9 their prototiles are equal. □

The corollary is constructive and it allows us to compute all prototiles of the Voronoi tiling of $\Lambda(\Omega)$ for a fixed $\Omega = [0, c)$:

Algorithm 5-12.

- Input: γ satisfying (F′), $\Omega = [0, c)$, L satisfying (5-11), e.g. given by (5-12).
- Output: The palette of $\Lambda(\Omega)$.

1. Compute the set $\Xi = \{\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = c\}$ given by (5-17).
2. For each interval $[\xi_j, \xi_{j+1})$ compute the corresponding L-patch.
3. Compute the corresponding prototiles to each of these patches.
4. Remove possible duplicates in the list of prototiles.



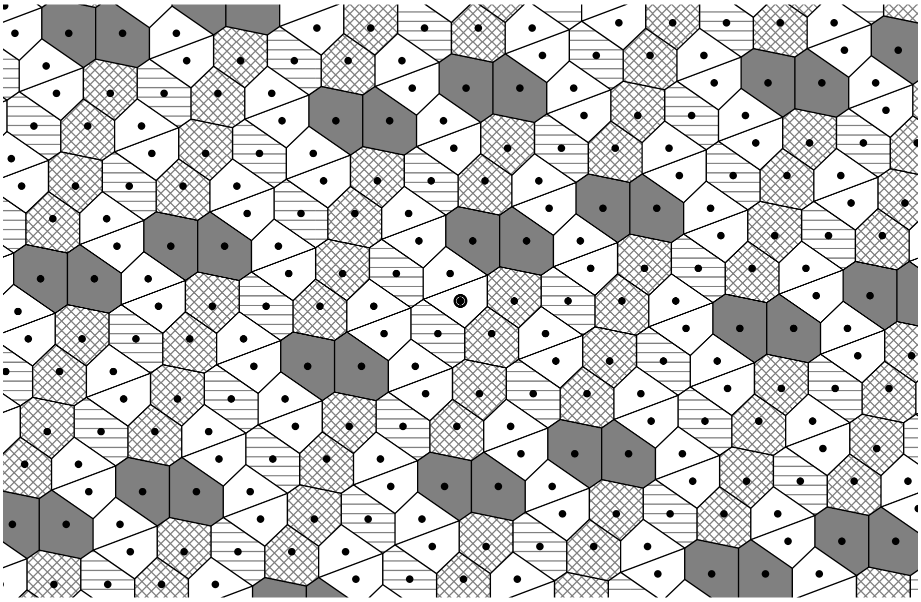
(Fig. 5-2) Voronoi prototiles (the palette) for $\mathcal{A}_2[\gamma] = \Lambda(\Omega)$, where $\Omega = [0, \frac{2}{1-\gamma'})$ and $\gamma = \gamma_t$ is the complex Tribonacci constant.

Example 5-13. We illustrate how the algorithm works for $\gamma = \gamma_t$ the complex Tribonacci constant and $c = 2/(1-\gamma') = \beta^2 + 1$, where we denote as usual $\beta := 1/\gamma'$. In this case, $\Lambda[0, c) = \mathcal{A}_2[\gamma]$ by Theorem 5-6. We have $k = -1$ in Lemma 5-8 and since $\arg\gamma \in (\pi/2, \pi)$, we have $p = 2$. Therefore L is the maximum of the values

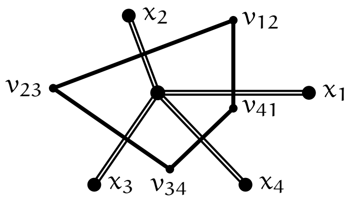
$$\frac{1}{|\gamma|}\left|\frac{\gamma(\gamma-1)}{\operatorname{Im}\gamma}\right| \approx 1.877, \quad \frac{1}{|\gamma|}\left|\frac{\gamma^2(\gamma^2-1)}{\operatorname{Im}(\gamma^2)}\right| \approx 1.877, \quad \frac{1}{|\gamma|}\left|\frac{\gamma^2(\gamma-1)}{\operatorname{Im}\gamma}\right| \approx 2.546,$$

i.e., $L = |\gamma(\gamma-1)|/\operatorname{Im}\gamma$. The set $\{ z' : z' \in \mathbb{Z}[\gamma'] \cap [0, c) \text{ and } |z| \leqslant L \}$ contains 28 points. The set Ξ, given as a union of two 28-element sets in (5-17), has only 33 elements instead of 56 because many elements appear in both of them. This gives 32 cases in steps 2–3 of the algorithm. After we remove the duplicates in the list of the 32 prototiles, we end up with the list in Figure 5-2. The double lines connect the center of the prototile with the centers of the neighboring tiles. A part of the Voronoi tiling of $\Lambda(\Omega)$ is drawn in Figure 5-3. Note that all computations are performed in the algebraic library of Sage [**Sage**]. Numbers $a + b\gamma + c\gamma^2 \in \mathbb{Z}[\gamma]$ are stored as triples of integers (a, b, c) and thus results of all arithmetic operations are precise.

Let us determine the parameters $\ell_2(\gamma)$ and $L_2(\gamma)$, with the help of relations (5-9). For each prototile $\mathcal{V}$, the value $\delta(\mathcal{V})$ is by definition the length of the shortest double line in the picture of $\mathcal{V}$. In Figure 5-4, the 1st prototile is depicted: the neighbors are (counterclockwise) $x_1 = 1$, $x_2 = 2 + 2\gamma + \gamma^2 = \gamma^{-2}$, $x_3 = 1 + \gamma + \gamma^2 = \gamma^{-1}$ and $x_4 = 2 + \gamma + \gamma^2 = 1 + \gamma^{-1}$. The closest point of these to 0 is $x_2 = \gamma^{-2}$. For the last prototile, the closest point is analogously $-\gamma^2$. Therefore $\delta(\mathcal{V}) = |\gamma^{-2}| = \gamma'$ for the first and the last prototile. For the rest of the prototiles, the closest point to 0 is $\pm(1 + \gamma + \gamma^2) = \pm\gamma^{-1}$, and therefore



(Fig. 5-3) Part of the Voronoi tiling of $\mathcal{A}_2[\gamma] = \Lambda(\Omega)$, where $\Omega = [0, \frac{2}{1-\gamma'})$ and $\gamma = \gamma_t$ is the complex Tribonacci constant. The point 0 is highlighted.



(Fig. 5-4) One of the prototiles of $\mathcal{A}_2[\gamma]$.

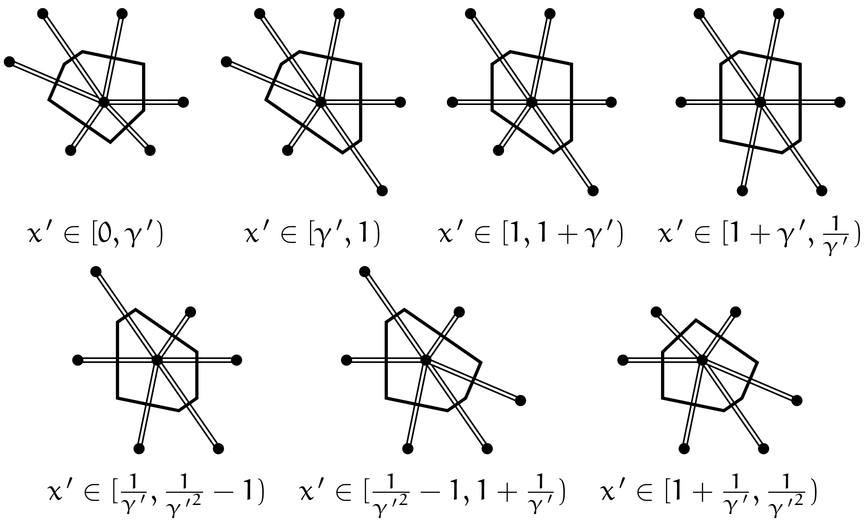
$\delta(\mathcal{V}) = |\gamma^{-1}| = \sqrt{\gamma'} = 1/\sqrt{\beta}$. Since $\ell_2(\gamma)$ is the minimum of all $\delta(\mathcal{V})$, we get that

$$\ell_2(\gamma) = \gamma' \approx 0.544.$$

To compute $L_2(\gamma)$, we first determine the value of $\Delta(\mathcal{V})$ for all prototiles. By definition, $\Delta(\mathcal{V})$ is twice the maximal distance from 0 to the vertices of $\mathcal{V}$. The vertices of the prototile are points v_{ij} satisfying that $|x_i - v_{ij}| = |x_j - v_{ij}| = |0 - v_{ij}|$, see Figure 5-4. This is the same condition as (5-13), thus the points v_{ij} are given by (5-14). Therefore we have

$$|v_{12}| = \frac{1}{2}\left|\frac{\gamma^{-2}(1-\gamma^{-2})}{\mathrm{Im}(\gamma^{-2})}\right| \approx 0.692, \quad |v_{23}| = \frac{1}{2}\left|\frac{\gamma^{-2}(1-\gamma^{-1})}{\mathrm{Im}(\gamma^{-1})}\right| \approx 0.692,$$
$$|v_{34}| = |v_{41}| = \frac{1}{2}\left|\frac{\gamma^{-1}(1+\gamma^{-1})}{\mathrm{Im}(\gamma^{-1})}\right| \approx 0.510.$$

Numerically, it seems that the first two values are equal. To see that this is true, we only have to check that $|1 + \gamma^{-1}| = 2|\mathrm{Re}(\gamma^{-1})|$, because $\frac{\mathrm{Im}(z^2)}{\mathrm{Im}\, z} = 2\,\mathrm{Re}\, z$ for any non-real $z \in \mathbb{C}$. Since γ^{-1} and $(\gamma^{-1})^\dagger$ are the Galois conjugates of β root of $X^3 = X^2 + X + 1$, we have $\gamma^{-1}(\gamma^{-1})^\dagger = 1/\beta$ and $\gamma^{-1} + (\gamma^{-1})^\dagger = 1 - \beta$ by Vieta's



(Fig. 5-5) Voronoi prototiles (the palette) for $\Lambda(\Omega)$, where $\Omega = [0, \frac{1}{\gamma'^2})$ and $\gamma = \gamma_t$ is the complex Tribonacci constant.

formulas. Now we easily verify that the numbers $|1+\gamma^{-1}|^2 = (1+\gamma^{-1})(1+(\gamma^{-1})^\dagger)$ and $4|\mathrm{Re}(\gamma^{-1})|^2 = (\gamma^{-1} + (\gamma^{-1})^\dagger)^2$ are equal. We can further simplify

$$|v_{23}|^2 = \frac{1}{4}\frac{\gamma^{-2}(\gamma^{-2})^\dagger(1-\gamma^{-1})(1-(\gamma^{-1})^\dagger)}{\left(\frac{1}{2i}(\gamma^{-1}-(\gamma^{-1})^\dagger)\right)^2} = \beta\frac{\beta^2-1}{3\beta^2-1},$$

because we see that the left-hand side is a symmetric rational function in γ^{-1}, $(\gamma^{-1})^\dagger$, therefore Vieta's formulas can be used to rewrite it in β's.

Whence, for the 1st prototile, the maximal distance is $\Delta(\mathcal{V}) = 2|v_{23}|$. It turns out that this is the value of $\Delta(\mathcal{V})$ for all the prototiles of $\Lambda(\Omega)$. Therefore $L_2(\gamma) = \Delta(\mathcal{V}(x))$ for all $x \in \mathcal{A}_2[\gamma]$ and the value is

$$L_2(\gamma) = 2\sqrt{\beta\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384.$$

Example 5-14. Let us give one more example. We fix the same $\gamma = \gamma_t$ as before and we take $c = (\gamma')^{-2} = \beta^2$. Then $p = 2$ and $k = 0$ satisfy the hypothesis of Lemma 5-8. Therefore

$$L = \left|\frac{\gamma^2(\gamma-1)}{\mathrm{Im}\,\gamma}\right| \approx 3.4531$$

satisfies (5-11). In this case, Ξ is of size 40. Figure 5-5 denotes the result of Algorithm 5-12. We get 7 different prototiles. The 4th one has $\delta(\mathcal{V}) = 1$, while all the other ones have $\delta(\mathcal{V}) = \sqrt{\gamma'}$. The value of $\Delta(\mathcal{V})$ is equal to $2\sqrt{\beta\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384$ for all of them.

We can now run Algorithm 5-12 again, using the better upper bound on $\Delta(\mathcal{V})$, namely $L \approx 1.384$. This can save us a lot of steps of the algorithm: The size of Ξ reduces from 40 to 8, so reduces the number of the steps. We will use this improved value of L in § 5-6, where we study the sets $\Lambda[0, c)$ for all $c > 0$.

In the two examples, we listed the palettes of $\Lambda[0,c)$ for two different values $c=\beta^2+1$ and $c=\beta^2$. Two prototiles appears in both lists. The natural question to ask is: For which values of c, a given prototile occurs in the palette of $\Lambda[0,c)$? Using Lemma 5-9, this question can be transformed to an easier one: for which values of c, a specific L-patch occurs in $\Lambda[0,c)$. Since we now treat L-patches for varying c, we denote them $\Pi^c_L(x)$, and for convenience we denote $\big(\Pi^c_L(x)\big)' := \{\, z' : z \in \Pi^c_L(x)\,\}$.

Lemma 5-15. *Let* $c_0 > 0$ *be fixed,* $c \in (0, c_0)$ *and* $L > 0$*. Denote* $-c_0 =: w_0 < w_1 < \cdots < w_{n-1} < w_n := c_0$ *the sequence of numbers such that*

$$W := \{w_1, \dots, w_{n-1}\} = \big\{\, z' \in \mathbb{Z}[\gamma'] : |z| \leqslant L \text{ and } z' \in (-c_0, c_0)\,\big\}. \tag{5-18}$$

Then

1. *For all* $x \in \Lambda[0,c)$ *we have*

$$\Pi^c_L(x) \subseteq \{\, z \in \mathbb{Z}[\gamma] : z' \in W\,\}.$$

2. *For all* $x \in \Lambda[0,c)$ *there exist* $i, k \in \mathbb{N}$, $1 \leqslant i \leqslant k \leqslant n-1$*, such that*

$$\{w_i, w_{i+1}, \dots, w_k\} = \big(\Pi^c_L(x)\big)'.$$

3. *Let* $1 \leqslant i \leqslant k \leqslant n-1$*. Then a finite set* $\{w_i, w_{i+1}, \dots, w_k\}$ *containing* 0 *equals* $(\Pi^c_L(x))'$ *for some* $x \in \Lambda[0,c)$ *if and only if*

$$w_k - w_i < c < w_{k+1} - w_{i-1}. \tag{5-19}$$

4. *For all* $x \in \Lambda[0,c)$ *there exists* $y \in \Lambda[0,c)$ *such that* $\Pi^c_L(y) = -\Pi^c_L(x)$.

Proof. 1. As $\Lambda[0,c) \subseteq \Lambda[0,c_0)$ we have $\Pi^c_L(x) \subseteq \Pi^{c_0}_L(x)$ and the statement follows from the relation (5-16).

2. Let i and k be the indices for which

$$w_i = \min\big(\Pi^c_L(x)\big)' \quad \text{and} \quad w_k = \max\big(\Pi^c_L(x)\big)'.$$

According to the relation (5-16) we get

$$0 \leqslant x' + w_i \quad \text{and} \quad x' + w_k < c. \tag{5-20}$$

Consider w_j for $j \in \mathbb{N}$, $i < j < k$. Then $w_i < w_j < w_k$, whence $0 \leqslant w_j + x' < c$. This implies that w_j belongs to $\big(\Pi^c_L(x)\big)'$ as well.

3. ($\Rightarrow$) Because of (5-20), we have $w_k - w_i < c$. Since w_{i-1} and w_{k+1} do not belong to $\big(\Pi_L^c(x)\big)'$, we have $x' + w_{i-1} < 0$ and $x' + w_{k+1} \geqslant c$. Hence $w_{k+1} - w_{i-1} > c$.
3. ($\Leftarrow$) Let $w_{i-1}, w_i, w_k, w_{k+1}$ satisfy (5-19). As $\mathbb{Z}[\gamma']$ is dense in $\mathbb{R}$, there exists $u \in (w_{i-1}, w_i)$ such that $u \in \mathbb{Z}[\gamma']$ and $u + c \in (w_k, w_{k+1})$. Put $x' := -u$. Then
$$x' + w_{i-1} < 0 < x' + w_i < x' + w_k < c < x' + w_{k+1}.$$
Since $w_i \leqslant 0 \leqslant w_k$, we have that $0 < x' < c$, therefore $x \in \Lambda[0, c)$. We conclude from item (2) that $\{w_i, w_{i+1}, \ldots, w_k\} = \big(\Pi_L^c(x)\big)'$.
4. Since W is a centrally symmetric set, i.e, $W = -W$, we have that $w_j = w_{n-j}$ for all $0 \leqslant j \leqslant n$. Then (5-19) is equivalent to
$$w_{n-i} - w_{n-k} < c < w_{n-i+1} - w_{n-k-1}.$$
According to item (3), the set $\{w_i, \ldots, w_k\}$ is an L-patch for some $x \in \Lambda[0, c)$ if and only if $\{-w_k, \ldots, -w_i\}$ is an L-patch for some $y \in \Lambda[0, c)$. □

Inequality (5-19) answers our question. To any L-patch, we can assign an open interval such that this patch occurs in $\Lambda[0, c)$ if and only if c lies in this interval. This fact has an important consequence: for any given set of L-patches, the range of c such that these patches are precisely the L-patches of $\Lambda[0, c)$ is an intersection of intervals and complements of intervals. As before, the result on L-patches implies the following result on palettes.

Corollary 5-16. *Let* $b_0, c_0 \in \mathbb{R}$ *satisfy that* $0 < b_0 < c_0$. *Denote by* $\mathrm{Pal}(\Omega)$ *the palette of* $\Lambda(\Omega)$*, i.e., the set of all prototiles of* $\Lambda(\Omega)$*. Then there exists a finite sequence* $b_0 =: \theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N := c_0$ *such that the mapping*
$$c \mapsto \mathrm{Pal}\big([0, c)\big)$$
is constant on each of the intervals (θ_{j-1}, θ_j) *for* $j = 1, \ldots, N$.

Proof. Consider L satisfying (5-11) for $\Lambda = \Lambda[0, b_0)$. For W given by (5-18) find $\theta_1 < \cdots < \theta_{N-1}$ such that
$$\Theta := (W - W) \cap (b_0, c_0) = \{\theta_1, \ldots, \theta_{N-1}\}. \tag{5-21}$$
Let $c, d \in (b_0, c_0)$ and suppose that the palette of $\Lambda[0, c)$ does not coincide with the palette of $\Lambda[0, d)$. Without loss of generality there exists an L-patch of $x \in \Lambda[0, c)$ that is not an L-patch of any $y \in \Lambda[0, d)$. This means that c satisfies inequalities (5-19) for some indices i, k, whereas d does not satisfy them. This fact implies that c and d are separated by a point $w_k - w_i \in W - W$. □

The previous corollary says that there exist only finitely many palettes for $\Lambda[0, c)$ with $c \in [b_0, c_0)$. The following algorithm determines them:

Algorithm 5-17.

- Input: γ satisfying (F'), $0 < b_0 < c_0$, L satisfying (5-11) for $\Omega = [0, b_0)$, e.g. given by (5-12).
- Output: All possible palettes $\mathrm{Pal}(\Omega)$ of $\Lambda(\Omega)$ for $\Omega = [0, c)$ and $b_0 \leqslant c < c_0$.

1. Compute the set $\Theta = \{\theta_1 < \cdots < \theta_{N-1}\}$ given by (5-21).
2. Using Algorithm 5-12, compute the palettes $\mathrm{Pal}(\Omega)$ for all $\Omega = [0, c)$ with $c = b_0, \frac{b_0+\theta_1}{2}, \theta_1, \ldots, \frac{\theta_{N-2}+\theta_{N-1}}{2}, \theta_{N-1}, \frac{\theta_{N-1}+c_0}{2}$.
3. Remove possible duplicates in the list of palettes.

In Corollary 5-16 and Algorithm 5-17, the assumption $b_0 > 0$ is very important, because there exist infinitely many $c \in (0, c_0)$ with different palettes. However, these palettes cannot differ too much. In fact, the self-similarity property (see Proposition 5-7) guarantees that the palette for the window $[0, \gamma' c)$ differs from the palette for $[0, c)$ only by a scaling factor γ. Therefore the knowledge of the palettes for $c \in [\gamma' c_0, c_0)$ is sufficient for the description of all palettes.

Remark 5-18. As a consequence of item (4) of Lemma 5-15, the list of L-patches for $\Lambda[0, c)$ is invariant under rotation by 180°. Therefore the palette $\mathrm{Pal}([0, c))$ is invariant as well. Figures 5-2 and 5-5 witness this phenomenon.

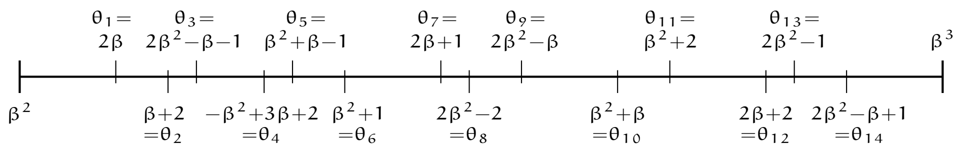
5-6 Complex Tribonacci case. Proof of Theorem 5-4

In this section, we describe the details of the proposed workflow on an example — the complex Tribonacci base $\gamma = \gamma_t$. We aim at the proof of Theorem 5-4. As usual, $\beta \coloneqq \gamma\gamma^\dagger = 1/\gamma'$. The theorem will be proved by combining the self-similarity property in Proposition 5-7 and the following result:

Proposition 5-19. *Let $\Omega = [0, c)$ with $c \in (\beta^2, \beta^3)$, where $\beta \coloneqq 1/\gamma'$ and γ is the complex Tribonacci constant. Denote $\Lambda \coloneqq \Lambda(\Omega)$. Then*

$$\min_{x\in\Lambda} \delta(\mathcal{V}(x)) = 1/\beta \quad \text{and} \quad \max_{x\in\Lambda} \Delta(\mathcal{V}(x)) = 2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}}. \tag{5-22}$$

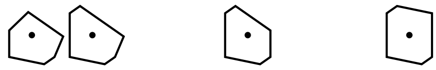
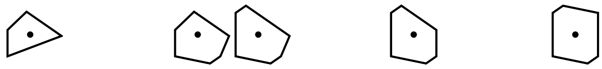
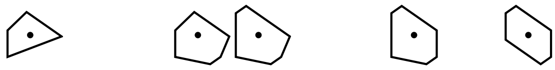
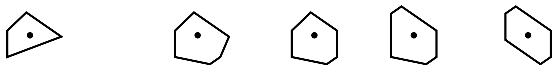
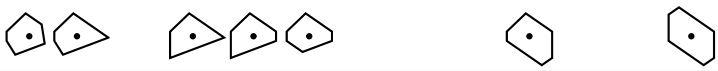
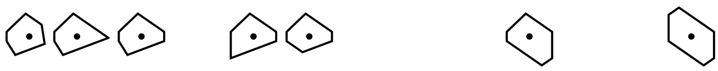
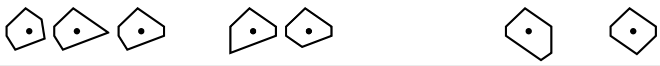
Proof. We put $b_0 \coloneqq \beta^2$ and $c_0 \coloneqq \beta^3$. In Example 5-14 we have shown that $L = 2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384$ satisfies (5-11) for $\Omega = [0, b_0)$. Using this L, we run Algorithm 5-17. The first step of the algorithm computes the set Θ defined by (5-21). This Θ has 14 elements, they are drawn in the following picture:



The number of cases in step 2 of the algorithm is then 30. This means that we have to run Algorithm 5-12 exactly 30 times to obtain all possible palettes. Amongst

TABLE 5-1

The prototiles for the complex Tribonacci constant for windows $\Omega = [0, c)$ with $c \in [\beta^2, \beta^3)$. We put $A := 2\sqrt{\frac{\beta^2-1}{3\beta^2-1}}$ and $B := A\sqrt{\beta}$. Each tile in the list appears rotated by 180° as well, we omit these to make the table shorter; see Remark 5-18. For a cut-point θ_i, the palette is the intersection of the palettes for the surrounding intervals, for instance $\mathrm{Pal}([0, \beta^2 + 1)) = \{\mathcal{V}_2, \mathcal{V}_6, \mathcal{V}_8, \mathcal{V}_9, -\mathcal{V}_8, -\mathcal{V}_6, -\mathcal{V}_2\}$.

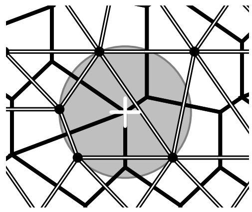
Interval for c	**The palette of** $\Lambda(\Omega)$**, where** $\Omega = [0, c)$
β^2	
$(\beta^2, 2\beta)$	
$(2\beta, \beta + 2)$	
$(\beta + 2, \beta^2 + 1)$	
$(\beta^2 + 1, 2\beta + 1)$	
$(2\beta + 1, \beta^2 + \beta)$	
$(\beta^2 + \beta, \beta^2 + 2)$	
$(\beta^2 + 2, 2\beta + 2)$	
$(2\beta + 2, \beta^3)$	

Tile	$\frac{1}{\gamma}\mathcal{V}_4$	$\mathcal{V}_1$	$\frac{1}{\gamma}\mathcal{V}_5$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\frac{1}{\gamma}\mathcal{V}_8$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\frac{1}{\gamma}\mathcal{V}_{10}$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
Value of δ	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	1
Value of Δ	A	B	A	B	B	A	B	B	B	B	B	A	B	B
Value of Δ^*	1	1	1	1	1	1	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	1	$\sqrt{\beta}$	$\sqrt{\beta}$

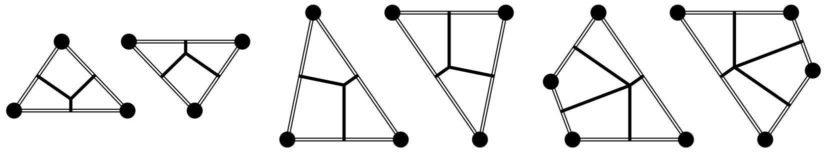
the 30 cases mentioned above, there are some duplicates, and we end up with only 16 cases: 8 cases correspond to cut-points $\theta_0, \theta_1, \theta_2, \theta_6, \theta_7, \theta_{10}, \theta_{11}, \theta_{12}$, the other 8 cases correspond to the open intervals between the cut-points. Moreover, we observe that for each cut-point θ_i, the palette $\mathrm{Pal}([0, \theta_i))$ is the intersection of the palettes of the two surrounding intervals. All the palettes for the intervals are depicted in Table 5-1.

At the bottom of the table, the values of $\delta(\mathcal{V})$ and $\Delta(\mathcal{V})$ are written out for each prototile. It turns out that every row of the table but the special case $c = \beta^2$ has the minimal value of δ equal to $1/\beta \approx 0.5437$ and the maximal value of Δ equal to $2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}} \approx 1.3843$. □

We recall that two of the runs of Algorithm 5-12, for $c = \frac{2}{1-\gamma'} = \beta^2 + 1 \in \Theta$, i.e., for $\mathcal{A}_2[\gamma]$, and for $c = \beta^2$ are explained in Examples 5-13 and 5-14 (cf. also



(Fig. 5-6) Part of Voronoi (in solid lines) and Delone (in double lines) tilings of $\mathcal{A}_2[\gamma]$ for $\gamma = \gamma_t$ the complex Tribonacci constant. The white cross is a vertex of the Voronoi tiling, and at the same time, it is the center of the gray circle, on which four points of $\mathcal{A}_2[\gamma]$ lie.



(Fig. 5-7) Delone tiles of the set $\mathcal{A}_2[\gamma]$, where $\gamma = \gamma_t$ is the complex Tribonacci constant.

Figures 5-2 and 5-5). We have drawn a part of the Voronoi tiling of $\mathcal{A}_2[\gamma]$ in Figure 5-3.

Proof of Theorem 5-4. The theorem is a direct corollary of Proposition 5-7, Theorem 5-6, Proposition 5-19 and of the following two facts:

- It cannot happen that $c = m/(1-\gamma') = (\gamma\gamma^\dagger)^k = \beta^k$ for some $m \geqslant 1$ and $k \in \mathbb{Z}$. For, assume on the contrary that the last equation holds. Then $\beta^k \geqslant m$ and so $k \geqslant 1$. Moreover, $k \geqslant 3$, since γ is cubic, and we have, by Galois isomorphism, that $m\gamma^k = 1 - \gamma$. The relation $|m\gamma^k| \geqslant |\gamma^3| > |1-\gamma|$ yields a contradiction.
- If $\mathcal{V}$ is a Voronoi prototile in $\Lambda(\Omega)$ then $\gamma^k\mathcal{V}$ is a prototile in $\gamma^k\Lambda(\Omega) = \Lambda((\gamma')^k\Omega)$ for any $k \in \mathbb{Z}$. For any $m \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ such that $(\gamma')^k \frac{m}{1-\gamma'} \in (\beta^2, \beta^3)$. □

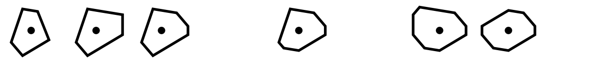
5-7 Delone tiling — dual to Voronoi tiling

From the Voronoi tiling we can construct its dual tiling: Let $\Lambda \subseteq \mathbb{C}$ be a Delone set. Consider a planar graph in $\mathbb{C}$ whose vertices are elements of the set Λ and edges are line segments connecting $x, y \in \Lambda$ where x and y are *neighbors*, i.e., their Voronoi tiles $\mathcal{V}(x)$ and $\mathcal{V}(y)$ share a side. This graph divides the complex plane into faces; these faces are called *Delone tiles*. The collection of Delone tiles is the *Delone tiling* of Λ.

All vertices of a Delone tile lie on a circle; its center is a vertex of the Voronoi tiling. This is illustrated in Figure 5-6, which shows a small part of the set $\mathcal{A}_2[\gamma]$, where γ is the complex Tribonacci constant; the quadrilateral is inscribed in the circle. The white cross marks the center of the circle and it is a common vertex of four Voronoi tiles.

The minimal distance $\inf_{x\in\Lambda} \delta(\mathcal{V}(x))$ is equal to the shortest edge in the Delone tiling. On the other hand, the longest edge in the Delone tiling is (in

TABLE 5-2
The prototiles for the complex Tribonacci constant for windows $\Omega = [0, c)$ with $c \in [\beta^5, \beta^6)$. We put $A := \frac{2}{\beta}\sqrt{\frac{\beta^2-1}{3-\beta}}$ and $B := A\sqrt{\beta}$. The remarks in Table 5-1 apply.

Interval for c	**The palette of** $\Lambda(\Omega)$**, where** $\Omega = [0, c)$								
β^5									
$(\beta^5, \beta^4 + \beta)$									
$(\beta^4 + \beta, 2\beta^3)$									
$(2\beta^3, \beta^4 + \beta^2)$									
$(\beta^4 + \beta^2, \beta^6)$									
Tile	$\frac{1}{\gamma}\mathcal{V}_1$	$\mathcal{V}_1$	$\frac{1}{\gamma}\mathcal{V}_2$	$\mathcal{V}_2$	$\frac{1}{\gamma}\mathcal{V}_3$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\frac{1}{\gamma}\mathcal{V}_5$	$\mathcal{V}_5$
Value of δ	$\frac{1}{\beta^2\sqrt{\beta}}$	$\frac{1}{\beta^2}$	$\frac{1}{\beta^2}$	$\frac{1}{\beta\sqrt{\beta}}$	$\frac{1}{\beta^2}$	$\frac{1}{\beta\sqrt{\beta}}$	$\frac{1}{\beta\sqrt{\beta}}$	$\frac{1}{\beta\sqrt{\beta}}$	$\frac{1}{\beta}$
Value of Δ	A	B	A	B	A	B	B	A	B
Value of Δ^*	$\frac{1}{\sqrt{\beta}}$	1	1	$\sqrt{\beta}$	1	$\sqrt{\beta}$	$\sqrt{\beta}$	1	$\sqrt{\beta}$

general) shorter than $\sup_{x\in\Lambda} \Delta(\mathcal{V}(x))$. Therefore, for a point $x \in \Lambda(\Omega)$ we can define

$$\Delta^*\big(\mathcal{V}(x)\big) := \max\big\{\,|x - y| : y \text{ is a neighbor of } x \text{ in } \Lambda\,\big\}$$

and study its maximum over all points $x \in \Lambda$.

We can apply this to the sets $\mathcal{A}_m[\gamma]$. We define

$$L^*_m(\gamma) := \sup_{x\in\mathcal{A}_m[\gamma]} \Delta^*\big(\mathcal{V}(x)\big)$$

if $\mathcal{A}_m[\gamma]$ is Delone, and $L^*_m(\gamma) = +\infty$ otherwise. When $\mathcal{A}_m[\gamma]$ is a cut-and-project set, we know that it has a finite local complexity and therefore finitely many different Delone tiles up to translation.

In the case of the complex Tribonacci base, the shapes of all Delone tiles of $\mathcal{A}_2[\gamma]$ are depicted in Figure 5-7. From Table 5-1 we get the following result:

Theorem 5-20. *With the hypothesis of Theorem 5-4, we have:*

$$L^*_m(\gamma) = |\gamma|^{3-k}.$$

5-8 MORE EXAMPLES

The procedure described in § 5-5, and used in § 5-6 on the complex Tribonacci constant γ_t, can be used for other numbers that satisfy Property (F′), for instance for the complex plastic constant $\gamma_p \approx -0.877 + 0.745i$ root of $X^3 + X^2 - 1 = 0$.

TABLE 5-3
List of all pairs (a, b) with $a \leqslant 200$ such that γ satisfies Property (F′) and the minimal distance between points of $\Lambda_\gamma[0, 1)$ is not $|\gamma|$, where $\gamma^3 + b\gamma^2 + a\gamma - 1 = 0$. In all cases, the minimal distance is the modulus of a unit in the field $\mathbb{Q}(\gamma)$.

a	b	**Minimal distance**	a	b	**Minimal distance**
1	2	$\lvert\gamma\rvert^{1/2}$	36	12	$\lvert\gamma\rvert^{1/2}$
2	−1	$\lvert\gamma\rvert^{1/2}$	49	14	$\lvert\gamma\rvert^{1/2}$
2	3	$\lvert\gamma\rvert^{1/4}$	64	16	$\lvert\gamma\rvert^{1/2}$
4	4	$\lvert\gamma\rvert^{1/2}$	81	18	$\lvert\gamma\rvert^{1/2}$
6	5	$\lvert\gamma\rvert^{1/5}$	100	20	$\lvert\gamma\rvert^{1/2}$
9	6	$\lvert\gamma\rvert^{1/2}$	121	22	$\lvert\gamma\rvert^{1/2}$
12	7	$\lvert\gamma\rvert^{1/9}$	144	24	$\lvert\gamma\rvert^{1/2}$
16	8	$\lvert\gamma\rvert^{1/2}$	169	26	$\lvert\gamma\rvert^{1/2}$
25	10	$\lvert\gamma\rvert^{1/2}$	196	28	$\lvert\gamma\rvert^{1/2}$

As the basic interval we choose the interval $[\beta^5, \beta^6)$, where $\beta = 1/\gamma_p' \approx 1.325$ is the minimal Pisot number. The result of Algorithm 5-17 is in Table 5-2. We get that when $c \in (\beta^5, \beta^6)$, the minimum of $\delta(\mathcal{V})$ is $\frac{1}{\beta^2\sqrt{\beta}}$ and the maximum of $\Delta(\mathcal{V})$ is $\frac{2}{\sqrt{\beta}}\sqrt{\frac{\beta^2-1}{3-\beta}}$. To determine $\ell_m(\gamma_p)$ and $L_m(\gamma_p)$ we distinguish two cases:

- For $c = \beta^5 = \frac{1}{1-\gamma_p'}$, i.e., for $\mathcal{A}_1[\gamma_p]$, we get that $\ell_1(\gamma_p) = 1/\beta^2$ and $L_1(\gamma_p) = \frac{2}{\sqrt{\beta}}\sqrt{\frac{\beta^2-1}{3-\beta}}$.
- For $m \geqslant 2$, we know that $\frac{m}{1-\gamma_p'}$ lies inside the open interval (β^k, β^{k+1}) for some $k \in \mathbb{Z}$. (Suppose this is not true. Then $\frac{m}{1-\gamma_p'} = \beta^k$ for some $m, k \in \mathbb{Z}$. But since $\frac{1}{1-\gamma_p'} = \beta^5$, we get that $m = \beta^{k-5} \in \mathbb{Z}$, which is a contradiction.)

Altogether we get the following result:

Theorem 5-21. *Let $\gamma = \gamma_p \approx -0.877 + 0.745i$ be a root of $X^3 + X^2 - 1 = 0$. Let γ_p' be the real Galois conjugate of γ_p. Then*

$$\ell_1(\gamma_p) = |\gamma_p|^{-4} \quad \textit{and} \quad L_1(\gamma_p) = 2\sqrt{\frac{1-(\gamma_p')^2}{3\gamma_p'-1}}. \tag{5-23}$$

For $m \geqslant 2$, let $k \in \mathbb{Z}$ be the greatest integer such that $m \geqslant (1-\gamma_p')\left(\frac{1}{\gamma_p'}\right)^k$. Then we have

$$\ell_m(\gamma_p) = |\gamma_p|^{-k} \quad \textit{and} \quad L_m(\gamma_p) = 2\sqrt{\frac{1-(\gamma_p')^2}{3\gamma_p'-1}}|\gamma_p|^{5-k}. \tag{5-24}$$

For both $\gamma = \gamma_t$ and γ_p, we get that $\ell_m(\gamma)$ is an integer power of $|\gamma|$. However, this property does not hold in general. Still, it seems to be a very "common" property with only some exception; all pairs (a, b) with $a \leqslant 200$, such that γ root of $X^3 + bX^2 + aX - 1$ has Property (F′) and $\ell_m(\gamma)$ is not an integer power of $|\gamma|$ for all m, are listed in Table 5-3.

5-9 Comments and open problems

This paper treated a family of cubic complex Pisot units γ — such ones that the real number $1/\gamma'$ is positive and satisfies Property (F). We used the concept of cut-and-project sets to study the properties of the sets $\mathcal{A}_m[\gamma]$. However, there are other cases where it might be possible to use this concept:

1. We can consider a different perspective of the Tribonacci constant. Let γ be the complex root of $Y^3 + Y^2 + Y - 1$, and put $\beta := 1/\gamma'$. Both γ and $-\gamma$ are complex Pisot units.

   It was shown by T. Vávra [**Váv14**] that the real Tribonacci constant β has the so-called Property ($-$F). Shortly speaking, all numbers from $\mathcal{I} \cap \mathbb{Z}[-1/\beta] = \mathcal{I} \cap \mathbb{Z}[\beta]$, where $\mathcal{I} := (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, have a finite expansion of the form $\frac{a_1}{-\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{-\beta^3} + \cdots$ with $a_j \in \{0, 1\}$. From this, we can show that $\mathcal{A}_m[-\gamma]$ is a cut-and-project set for arbitrary $m \geqslant 1$. The idea goes along the lines of the proof of Theorem 5-6. Recently [**Krč15**, **KV15**], a full description of cubic units with Property ($-$F) has been given.

2. Consider any real Pisot unit β of degree n. Let $\gamma = i\sqrt{\beta}$. Then γ is a complex Pisot unit of degree $2n$, its Galois conjugates are $\gamma^\dagger$ and $\pm i\sqrt{\beta'}$ for β' conjugates of β.

   Clearly $\mathcal{A}_m[\gamma] = \mathcal{A}_m[-\beta] + i\sqrt{\beta}\mathcal{A}_m[-\beta]$. Therefore the Voronoi tiles of $\mathcal{A}_m[\gamma]$ are rectangles. Values $\ell_m(\gamma)$ and $L_m(\gamma)$ can be easily obtained from the minimal and maximal distances in $\mathcal{A}_m[-\beta]$. In the case $n = 2$, relations between $\mathcal{A}_m[-\beta]$ and cut-and-project sets in dimensions $d = e = 1$ were established in [**MPP15**], implying that $\mathcal{A}_m[\gamma]$ is related to cut-and-project sets in dimensions $d = e = 2$.

   Let us note that Zaïmi [**Zaï04**] evaluated $\ell_m(\gamma)$ for $\gamma = i\sqrt{\beta}$, $m = \lfloor \beta^2 \rfloor$ and $\beta > 1$ the root of $Y^2 - aY - a$, $a \in \mathbb{N}$.

3. In the cubic case, we can weaken the hypothesis of Theorem 5-6. For a fixed m, Property (F) can be replaced by the assumption that all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have a finite β-representation over the alphabet $\{0, 1, \ldots, m\}$, where we denote $\beta := 1/\gamma' > 1$. Under such an assumption, $\mathcal{A}_m[\gamma]$ is a cut-and-project set.

   Akiyama, Rao and Steiner [**ARS04**] described precisely the set of purely periodic expansions of points from $\mathbb{Z}[\beta]$. They have shown that all of them are of the form $\bullet ccc\cdots = \bullet c^\omega$, where $0 \leqslant c < \lfloor \beta \rfloor$ and $(a + b) \mid c$. Since all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have finite or periodic β-expansions [**Sch80**] (and the only periods are therefore the ones mentioned above), it is satisfactory to find m_1 such that the number $\bullet(a+b)^\omega$ has a finite representation over the alphabet $\{0, \ldots, m_1\}$. Under this hypothesis, all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have a finite representation over the alphabet $\{0, \ldots, m\}$ for all $m \geqslant m_1 \lfloor \frac{\beta}{a+b} \rfloor$. We were not able to establish the hypothesis in all cases. We list some cases in Table 5-4.

TABLE 5-4
List of pairs of a, b such that $\mathcal{A}_m[\gamma]$ is a cut-and-project set, where γ is the non-real root of $Y^3 + bY^2 + aY - 1$ and $m \geqslant m_1 \lfloor \frac{1/\gamma'}{a+b} \rfloor$.

b	a	m_1	Representation of $\bullet(a{+}b)^\omega$
-2	$\geqslant 3$	$2a-2$	$\bullet(a{-}3)(2a{-}2)(a{-}3)(0)(1)$
-3	$\geqslant 7$	$3a-6$	$\bullet(a{-}4)(2a{-}5)(3a{-}6)(a{-}7)(0)(1)$
	$=6$	10	$\bullet(2)(7)(10)(10)(0)(0)(1)$
	$=5$	9	$\bullet(0)(9)(9)(5)(0)(0)(1)$
	$=4$	7	$\bullet(0)(2)(6)(7)(0)^3(1)$
-4	$\geqslant 8$	$8a-11$	$\bullet(a{-}5)(2a{-}11)(8a{-}11)(4a{-}31)(a{-}8)(0)(1)$
	$=7$	39	$\bullet(0)(16)(39)(27)(0)^3(1)$
	$=6$	47	$\bullet(0)(3)(44)(47)(0)^4(1)$

4. Quartic Pisot units γ with $|\gamma| \in (1,2)$ are treated by Dombek, Masáková and Ziegler in [**DMZ15**]. The authors study the question of whether every element of the ring $\mathbb{Z}[\gamma]$ of integers of $\mathbb{Q}(\gamma)$ can be written as a sum of distinct units. If the only units on the unit circle are ± 1, then the question can be interpreted as Property (F) over the alphabet $\{-1, 0, 1\}$. Therefore the concept of cut-and-project sets can be applied to these quartic bases and symmetric alphabets as well.

Let us conclude with several open questions:

Problem 5a. Is it true that all real cubic Pisot units β with a complex conjugate satisfy the following: There exists $m \in \mathbb{N}$ such that all numbers from $\mathbb{Z}[\beta] \cap [0,1)$ have a finite β-representation over the alphabet $\{0, \dots, m\}$?

Problem 5b. Answer "yes" or "no": For all cubic units γ that satisfy Property (F$'$) and for all $m \geqslant |\gamma|^2 - 1$ we have that $\ell_m(\gamma)$ is the modulus of a unit in $\mathbb{Z}[\gamma]$.

Problem 5c. Answer "yes" or "no": For all cubic units γ that satisfy Property (F$'$) but for roots of $X^3 + bX^2 + aX - 1$ with $(a, b) \in \{(2,-1), (2,3), (6,5), (12,7)\} \cup \{ (k^2, 2k) : k \in \mathbb{N},\ k \geqslant 1 \}$, we have that $\ell_m(\gamma)$ is an integer power of $|\gamma|$ for all m.

Problem 5d. Which Pisot numbers β satisfy that there exists $m \in \mathbb{N}$ such that all $x \in \mathbb{Z}[\beta]$ have a finite β-representation over the alphabet $\{0, 1, \dots, m\}$?

Problem 5e. It is well known that, in the real case, $\mathcal{A}_m[\beta]$ is a relatively dense set in $\mathbb{R}_+$ if and only if $m > \beta - 1$. Can we state analogous result in the complex case? In particular, is $\mathcal{A}_m[\gamma]$ relatively dense set in $\mathbb{C}$ for all $m > |\gamma|^2 - 1$?

Can the complex modification of the Feng's result [**Fen15**] be proved, namely that $\ell_m(\gamma) = 0$ if and only if $m > |\gamma|^2 - 1$ and γ is not a complex Pisot number?

APPENDIX A

Index of Notation

Symbol	**Meaning**
$\mathcal{A}_m$	The alphabet $\{0, 1, \ldots, m\}$
$\mathcal{A}[\beta]$	Spectrum of β with digits in $\mathcal{A}$
$B_r(x)$	Ball of diameter r centered at x
$\mathbb{C}$	Complex numbers
x_f	Embedding of $x \in \mathbb{Q}(\beta)$ in finite places of $\mathbb{Q}(\beta)$ that divide β
$\mathfrak{h}(x)$	Hensel β-expansion of $x \in \mathbb{Z}[\beta]$
H-lim	Hausdorff limit
i	Imaginary unit
$\operatorname{Im} z$	Imaginary part of complex number z
K_f	Completion of field K w.r.t. finite places that divide β
$\ell_m(\gamma)$	Minimal distance between points of spectrum $\mathcal{A}_m[\beta]$
$L_m(\gamma)$	Maximal diameter of a ball that does not meet $\mathcal{A}_m[\beta]$
$L^*_m(\gamma)$	Maximal distance between neighbours in $\mathcal{A}_m[\beta]$
$N(x)$	Norm of algebraic integer x
$\mathbb{N}$	Natural numbers, including 0
$\mathcal{O}_K$	Ring of integers of field K
P_β	Minimal polynomial of algebraic number β
$\operatorname{Pref}_n w$	Prefix of word w of length n
$\mathbb{Q}$	Rational numbers
$\mathbb{Q}(\beta)$	Field extension of $\mathbb{Q}$ by β
$\mathcal{Q}(x)$	Rauzy fractal, $\mathcal{Q}(x) = \text{H-lim}\, \Psi(x - \beta^n T^{-n}(x))$
$\mathcal{Q}_f(x)$	Rauzy fractal, $\mathcal{Q}_f(x) = \text{H-lim}\, \Psi_f(x - \beta^n T^{-n}(x))$
$\mathbb{R}$	Real numbers
$\mathcal{R}(x)$	Rauzy fractal, $\mathcal{R}(x) = \text{H-lim}\, \Psi(\beta^n T^{-n}(x))$ for $x \in \mathbb{Z}[\beta]$
$\operatorname{Re} z$	Real part of complex number z
T_G, T_S, T_B	Greedy, symmetric and balanced β-transformations
$\mathcal{T}$	Natural extension mapping
$\mathcal{V}(x)$	Voronoi tile of x
$\mathcal{X}$	Natural extension domain
$\mathbb{Z}$	Integers
$\mathbb{Z}[X]$	Ring of polynomials with integer coefficients
$\mathbb{Z}[\beta]$	Ring of integer combinations of non-negative powers of β

Symbol	Meaning
γ_t	Complex Tribonacci constant, the root of $X^3 = -X^2 - X + 1$ such that $\operatorname{Im}\gamma_t > 0$
γ_p	Complex plastic constant, the root of $X^3 = -X^2 + 1$ such that $\operatorname{Im}\gamma_p > 0$
$\delta(\mathcal{V})$	Distance between center of Voronoi tile $\mathcal{V}$ and its nearest neighbour
$\Delta(\mathcal{V})$	Distance between center of $\mathcal{V}$ and its furthest vertex
$\Delta^*(\mathcal{V})$	Distance between center of $\mathcal{V}$ and its furthest neighbour
ε	Empty word
$\Lambda_\beta(\Omega)$	Model set associated to Pisot number β
$\Pi_\rho(x)$	Patch (of model set) of size ρ around point x
φ_g	Golden ratio, the positive root of $X^2 = X + 1$
φ_p	Minimal Pisot number (or plastic constant), the real root of $X^3 = X + 1$
φ_t	Tribonacci constant, the real root of $X^3 = X^2 + X + 1$
$\psi_{(j)}$	jth Galois isomorphism of $\mathbb{Q}(\beta)$
Ψ	Direct product of non-identity Galois isomorphisms of $\mathbb{Q}(\beta)$
Ψ_0	Direct product of all Galois isomorphisms of $\mathbb{Q}(\beta)$
Ψ_f	Direct product $\Psi_f(x) = (\Psi(x), x_f)$
$\Psi_{0,f}$	Direct product $\Psi_{0,f}(x) = (\Psi_0(x), x_f)$
w^ω	Periodic word produced by infinite repetition of w, $www\cdots$
$z^\dagger$	Complex conjugate of z
$\#S$	Number of elements of set S, $\#S \in \mathbb{N} \cup \{\infty\}$
$\langle\cdot;\cdot\rangle$	Polynomial representation of a word, $\langle u_0 u_1 \cdots ; X\rangle = \sum_k u_k X^k$
$a \perp b$	Relation of being co-prime, i.e., $\gcd(a, b) = 1$
w^*	Kleene star operation, the language $w^* = \{\varepsilon, w, ww, w^3, \cdots\}$

APPENDIX B

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