MULTIPLE TILINGS ASSOCIATED TO d-BONACCI BETA-EXPANSIONS

TOMÁŠ HEJDA

Abstract. It is a well-known fact that when $\beta > 1$ is a d-Bonacci number, i.e., $\beta^d = \beta^{d-1} + \beta^{d-2} + \cdots + \beta + 1$ for some $d \geq 2$, then the Rauzy fractals arising in the greedy β-expansions tile the space $\mathbb{R}^{d-1}$. However, it was recently shown that the Rauzy fractals arising in the symmetric Tribonacci expansions form a multiple tiling with covering degree 2, i.e., almost every point of $\mathbb{R}^2$ lies in exactly 2 tiles. We show that the covering degree for symmetric d-Bonacci expansions is equal to $d-1$ for any d. We moreover characterize which tiles lie in the same layer of the multiple tiling.

1. Introduction

Tilings arising from β-expansions were first studied in the 1980s by A. Rauzy [Rau82] and W. Thurston [Thu89]. They consider the greedy β-expansions that are associated to the transformation $T_{\mathsf{G}} \colon x \mapsto \beta x - \lfloor \beta x \rfloor$. S. Akiyama [Aki02] showed that the collection of β-tiles forms a tiling if and only if β satisfies the so-called *weak finiteness property* (W). It is conjectured that all Pisot numbers satisfy (W) for the greedy expansions, this is one of the versions of the famous unresolved *Pisot conjecture* [Hol96, ARS04].

If we drop the "greedy" hypothesis, things are getting more interesting. C. Kalle and W. Steiner [KS12] showed that the symmetric β-expansions for two particular cubic Pisot numbers β induce a double tiling — i.e., a multiple tiling such that almost every point of the tiled space lies in exactly two tiles. More generally, they proved that every "well-behaving" β-transformation with a Pisot number β induces a multiple tiling. Multiple tilings in various related settings were as well considered by S. Ito and H. Rao [IR06].

In this paper we concentrate on the symmetric β-expansions associated to the transformation $T_{\mathsf{S}} \colon x \mapsto \beta x - \lfloor \beta x - \frac{1}{2} \rfloor$; we define T_{S} on two intervals $[-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$ that form the support of its invariant measure, cf. Lemma 1. This transformation was studied before e.g. by S. Akiyama and K. Scheicher in the context of shift radix systems [AS07]. We show the following theorem about the multiple tiling:

Theorem 1. *Let $d \in \mathbb{N}$, $d \geq 2$, and let $\beta \in (1,2)$ be the d-Bonacci number, i.e., the Pisot number satisfying $\beta^d = \beta^{d-1} + \cdots + \beta + 1$. Then the symmetric β-expansions induce a multiple tiling of $\mathbb{R}^{d-1}$ with covering degree equal to $d-1$.*

(Note that for any particular β and any particular transformation, the degree of the multiple tiling can be computed from the intersection [or boundary] graph, eventually multi-graph, as defined for instance by A. Siegel and J. Thuswaldner [ST09]; however, such an algorithmic approach is not usable for an infinite number of cases.)

We also characterize the tiles that form the distinct layers of the multiple tiling:

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Theorem 2. *Let $d \in \mathbb{N}$, $d \geq 3$, and let $\beta \in (1,2)$ be the d-Bonacci number. Let $h \in \{1,2,\dots,d-1\}$. Then the collection of tiles $\{\, \mathcal{R}(x) : x \in \mathcal{L}_h \,\}$, where*

$$\mathcal{L}_h := \Big([\![h]\!] \cap \big[1-\tfrac{\beta}{2}, \tfrac{1}{2}\big) \Big) \cup \Big([\![h-d]\!] \cap \big[-\tfrac{1}{2}, \tfrac{\beta}{2}-1\big) \Big), \tag{1.1}$$

forms a tiling of $\mathbb{R}^{d-1}$, that is, it is a layer of the multiple tiling guaranteed by Theorem 1. Here we denote $[\![j]\!] := j + (\beta-1)\mathbb{Z}[\beta]$.

The two results rely substantially on the knowledge of the purely periodic integer points of T_{S}:

Theorem 3. *Let $d \in \mathbb{N}$, $d \geq 2$, and let $\beta \in (1,2)$ be the d-Bonacci number. Let $\mathcal{P}$ denote the set of non-zero $x \in \mathbb{Z}[\beta]$ such that $T_{\mathsf{S}}^p x = x$ for some $p \geq 1$. Then*

$$\mathcal{P} \cup \{0\} = \big\{ \pm \bullet 0 p_2 p_3 \cdots p_d : p_i \in \{0,1\} \big\} = \Big\{ \pm \sum_{i=2}^{d} p_i \beta^{-i} : p_i \in \{0,1\} \Big\}.$$

(We exclude 0 from $\mathcal{P}$ as it does not lie in the support of the invariant measure of T_{S}.)

The paper is organized as follows. In the following section we define all the necessary notions. The theorems are proved in Section 3. We conclude by a pair of related open questions in Section 4.

2. Preliminaries

2.1. **Pisot numbers.** An algebraic integer $\beta > 1$ is a *Pisot number* if all its Galois conjugates, i.e., the other roots of its minimal polynomial, lie inside of the unit complex circle. As usual, $\mathbb{Z}[\beta]$ denotes the ring of integer combinations of powers of β, and $\mathbb{Q}(\beta)$ denotes the field generated by the rational numbers and by β.

Suppose that β is of degree d and has $2e < d$ complex Galois conjugates $\beta_{(1)},\dots,\beta_{(e)}$, $\beta^\star_{(1)},\dots,\beta^\star_{(e)}$ and $d-2e-1$ real ones $\beta_{(e+1)},\dots,\beta_{(d-e-1)}$. Denote $\sigma_{(j)} \colon \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_{(j)})$ the corresponding Galois isomorphisms. Then we put

$$\Phi \colon \mathbb{Q}(\beta) \to \prod_{j=1}^{d-e-1} \mathbb{Q}(\beta_{(j)}), \quad x \mapsto \big(\sigma_{(1)}(x),\dots,\sigma_{(d-e-1)}(x)\big).$$

Since $\prod_{j=1}^{d-e-1} \mathbb{Q}(\beta_{(j)}) \subset \mathbb{C}^e \times \mathbb{R}^{d-2e-1} \simeq \mathbb{R}^{d-1}$, we consider that $\Phi \colon \mathbb{Q}(\beta) \to \mathbb{R}^{d-1}$. We have the closure properties $\overline{\Phi(\mathbb{Z}[\beta])} = \overline{\Phi(\mathbb{Q}(\beta))} = \mathbb{R}^{d-1}$.

In this paper, we focus on d-Bonacci numbers. For $d \geq 2$ a *d-Bonacci number* is the Pisot root of the polynomial $\beta^d = \beta^{d-1} + \cdots + \beta + 1$.

We say that two numbers $x, y \in \mathbb{Z}[\beta]$ are *congruent modulo* $\beta - 1$ if $y - x \in (\beta-1)\mathbb{Z}[\beta]$. By $[\![h]\!]$, for $h \in \mathbb{Z}[\beta]$, we denote the congruence class modulo $\beta-1$ that contains h, i.e., $[\![h]\!] := h + (\beta-1)\mathbb{Z}[\beta]$. If β is a d-Bonacci number, then the norm of $\beta-1$ is $N(\beta-1) = -(d-1)$. Therefore there are exactly $d-1$ distinct classes modulo $\beta-1$ and we can take numbers $h \in \{1,2,\dots,d-1\}$ as their representatives, i.e.,

$$\mathbb{Z}[\beta] = \bigcup_{h=1}^{d-1} [\![h]\!] = \bigcup_{h=1}^{d-1} h + (\beta-1)\mathbb{Z}[\beta].$$

2.2. **β-expansions.** We fix $\beta \in (1,2)$. Let $X \subset \mathbb{R}$ be union of intervals and $D\colon X \mapsto \mathbb{Z}$ be a piecewise constant function (*digit function*) such that $\beta x - D(x) \in X$ for all $x \in X$. Then the map $T\colon X \to X$, $x \mapsto \beta x - D(x)$ is a β-transformation. The β-expansion of $x \in X$ is then the (right-infinite) sequence $x_1x_2x_3\cdots \in (D(X))^\omega$, where $x_i = DT^{i-1}x$. We say that $x_1x_2x_3\cdots \in \mathbb{Z}^\omega$ is *T-admissible* if it is the expansion of some $x \in X$.

We define two particular β-transformations:

(1) Let $X_{\mathsf{S}} \coloneqq [-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$ and $D_{\mathsf{S}}(x) \coloneqq \lfloor \beta x - \frac{1}{2} \rceil \in \{\bar{1}, 0, 1\}$ (we denote $\overline{a} \coloneqq -a$ for convenience). This defines the *symmetric β-expansions.* We denote T_{S} the transformation and $(x)_{\mathsf{S}} \in \{\bar{1}, 0, 1\}^\omega$ the expansion of $x \in X_{\mathsf{S}}$.

(2) Let $X_{\mathsf{B}} \coloneqq [\frac{2-\beta}{2\beta-2}, \frac{\beta}{2\beta-2})$ and $D_{\mathsf{B}}(x) \coloneqq 1$ if $x \geq \frac{1}{2\beta-2}$ and $D_{\mathsf{B}}(x) \coloneqq 0$ otherwise. This defines the *balanced β-expansions.* We denote T_{B} and $(x)_{\mathsf{B}} \in \{0,1\}^\omega$ accordingly.

Both T_{S} and T_{B} are plotted in Figure 1 for the Tribonacci number.

Besides expansions, we consider arbitrary representations. Any bounded sequence of integers $x_{-N}\cdots x_{-1}x_{0\bullet}x_1x_2\cdots$ is a *representation* of $x = \sum_{i \geq -N} x_i\beta^{-i} \in \mathbb{R}$.

A *factor* of a sequence $x_1x_2x_3\cdots$ is any finite word $x_kx_{k+1}\cdots x_{l-1}$ with $l \geq k \geq -N$. A sequence $x_1x_2\cdots$ is *periodic* if $(\exists k, p \in \mathbb{N}, p \geq 1)(\forall i > k)(x_{i+p} = x_i)$. It is *purely periodic* if $k = 0$.

2.3. **Rauzy fractals.** We consider the symmetric β-transformations for Pisot units β. The symmetric β-transformation T_{S} possesses a unique invariant measure absolutely continuous w.r.t. the Lebesgue measure. For any $x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}$, we define the *$\beta$-tile* (or *Rauzy fractal*) as the Hausdorff limit

$$\mathcal{R}(x) \coloneqq \lim_{n\to\infty} \Phi\big(\beta^n T^{-n}(x)\big) \subset \mathbb{R}^{d-1}.$$

Note that $T^{-n}(-x) = -T^{-n}(x)$ for all $x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}$ and all n, therefore $\mathcal{R}(-x) = -\mathcal{R}(x)$.

The Rauzy fractals induce a multiple tiling, as was shown in Theorem 4.10 of [KS12]. We recall that the family of tiles $\mathcal{T} \coloneqq \{\mathcal{R}(x)\}_{x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}}$ is a *multiple tiling* if the following is satisfied:

(1) The tiles $\mathcal{R}(x)$ take only finitely many shapes (i.e., are only finitely many modulo translation in $\mathbb{R}^{d-1}$).
(2) The family $\mathcal{T}$ is locally finite, i.e., for every bounded set $U \subset \mathbb{R}^{d-1}$, only finitely many tiles from $\mathcal{T}$ intersect U.
(3) The family $\mathcal{T}$ covers $\mathbb{R}^{d-1}$, i.e., for every $y \in \mathbb{R}^{d-1}$ there exists $\mathcal{R}(x) \in \mathcal{T}$ such that $y \in \mathcal{R}(x)$.
(4) Every tile $\mathcal{R}(x)$ is a closure of its interior.
(5) There exists an integer $m \geq 1$ such that almost every point in $\mathbb{R}^{d-1}$ lies in exactly m tiles from $\mathcal{T}$; this m is called the *covering degree* of $\mathcal{T}$.

If $m = 1$, we say that $\mathcal{T}$ is a *tiling.* Every multiple tiling with covering degree $m \geq 2$ is a union of m tilings; we call these tilings *layers* of the multiple tiling.

3. Proofs

From now on, we suppose that $d \geq 3$ is an integer and $\beta > 1$ is the d-Bonacci number, i.e., the root of $\beta^d = \beta^{d-1} + \beta^{d-2} + \cdots + \beta + 1$. In Lemma 1 we show that the support of the invariant measure of T_{S} is the whole X_{S}; from this, we conclude that $\{\mathcal{R}(x)\}_{x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}}$ is a multiple tiling [KS12, Theorem 4.10]. In Lemma 2 we establish a strong relation between the symmetric and the balanced expansion. This allows us to use arithmetic results on balanced expansions in Lemmas 3 and 4 to determine the degree of the multiple

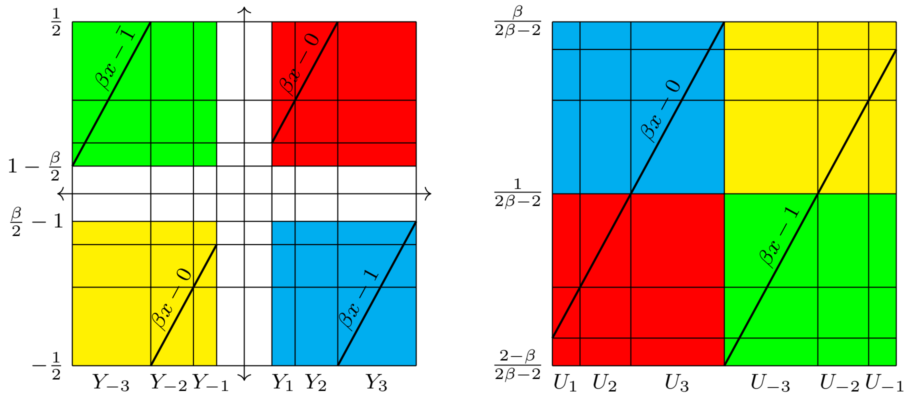

FIGURE 1. Transformations T_{S} (left) and T_{B} (right) for $d=3$.

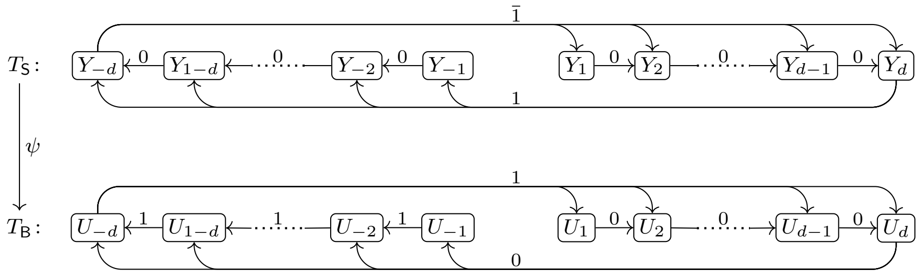

FIGURE 2. The automata accepting the T_{S}-admissible sequences (top) and the T_{B}-admissible ones (bottom).

tiling, which is done in Lemmas 5, 6 and 7. The proof of Theorem 3 is given after Lemma 4, the proofs of Theorems 1 and 2 are at the very end of the section.

Lemma 1. *The support of the invariant measure of T_{S} is the whole domain $X_{\mathsf{S}} = [-\frac{1}{2}, \frac{\beta}{2}-1) \cup [1-\frac{\beta}{2}, \frac{1}{2})$.*

Proof. Denote $l \coloneqq -\frac{1}{2}$. Put $Y_d \coloneqq [T_{\mathsf{S}}^d l, -l)$ and $Y_k \coloneqq [T_{\mathsf{S}}^k l, T_{\mathsf{S}}^{k+1} l)$ for $1 \le k \le d-1$. Similarly, put $Y_{-d} \coloneqq [l, -T_{\mathsf{S}}^d l)$ and $Y_{-k} \coloneqq [-T_{\mathsf{S}}^{k+1} l, -T_{\mathsf{S}}^k l)$ for $1 \le k \le d-1$, see Figure 1.

Define a measure μ by

$$\mathrm{d}\mu(x) = f(x)\,\mathrm{d}x \coloneqq \Big(\frac{1}{\beta} + \frac{1}{\beta^2} + \cdots + \frac{1}{\beta^k}\Big)\mathrm{d}x \quad \text{for } x \in Y_{\pm k},\ 1 \le k \le d.$$

Then we verify that for any $x \in X_{\mathsf{S}}$, we have

$$\mu\big([x, x+\mathrm{d}x)\big) = f(x)\,\mathrm{d}x = \frac{1}{\beta}\,\mathrm{d}x \sum_{\substack{y \in X_{\mathsf{S}} \\ T_{\mathsf{S}} y = x}} f(y) = \mu\big(T_{\mathsf{S}}^{-1}[x, x+\mathrm{d}x)\big),$$

because

$$T_{\mathsf{S}} Y_{\pm k} = \begin{cases} Y_{\mp 1} \cup Y_{\mp 2} \cup \cdots \cup Y_{\mp d} & \text{if } k = d; \\ Y_{\pm(k+1)} & \text{otherwise.} \end{cases} \tag{3.1}$$

Therefore μ is the invariant measure of T_{S}. ☐

Lemma 2. *Let $x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}$. Define*

$$\psi\colon X_{\mathsf{S}} \to X_{\mathsf{B}}, \quad x \mapsto \begin{cases} \frac{1}{\beta-1}x & \text{if } x \in [1-\frac{\beta}{2}, \frac{1}{2}); \\ \frac{1}{\beta-1}(x+1) & \text{if } x \in [-\frac{1}{2}, \frac{\beta}{2}-1). \end{cases}$$

Suppose that $(\psi x)_{\mathsf{B}} = t_1t_2t_3\cdots$. Then $(x)_{\mathsf{S}} = (t_2-t_1)(t_3-t_2)(t_4-t_3)\cdots$. Moreover, $(x)_{\mathsf{S}}$ is purely periodic if and only if $(\psi x)_{\mathsf{B}}$ is, and the length of the periods is the same.

Proof. The transformations T_{S} and T_{B} are conjugated via ψ, i.e., the following diagram commutes:

$$\begin{array}{ccc} X_{\mathsf{S}} & \xrightarrow{T_{\mathsf{S}}} & X_{\mathsf{S}} \\ \cong\big\downarrow\psi & & \cong\big\downarrow\psi \\ X_{\mathsf{B}} & \xrightarrow{T_{\mathsf{B}}} & X_{\mathsf{B}} \end{array}$$

(see Figure 1). If we denote $U_{\pm k} := \psi Y_{\pm k}$, we get that (3.1) is true for the sets U_k as well. We depict the acceptance automaton for balanced expansions in Figure 2 bottom. If an infinite path in the bottom automaton is labelled by $t_1t_2t_3\cdots$, then the corresponding path in the top automaton is labelled by $x_1x_2x_3\cdots$ with $x_i = t_{i+1} - t_i$.

The periodicity is preserved because $T_{\mathsf{S}}^p x = x \iff T_{\mathsf{B}}^p \psi x = \psi x$. □

Lemma 3. *Suppose that the balanced expansion of $x \in \mathbb{Q}(\beta) \cap X_{\mathsf{B}}$ has the form*

$$(x)_{\mathsf{B}} = x_1x_2x_3\cdots x_n(x_{n+1}\cdots x_{n+d})^{\omega}.$$

Then for any $z \in \mathbb{Z}[\beta]$ such that $x + z \in X_{\mathsf{B}}$, the balanced expansion of $x + z$ has the form

$$(x+z)_{\mathsf{B}} = y_1y_2y_3\cdots y_m(y_{m+1}\cdots y_{m+d})^{\omega},$$

where, moreover, $x_{n+1} + \cdots + x_{n+d} = y_{m+1} + \cdots + y_{m+d}$.

Proof. Clearly it is enough to consider the simplest case $z = \pm\beta^{-k}$ for some $k \geq 2$, since any $z \in \mathbb{Z}[\beta]$ is a finite sum of powers of β. Then $x + z = {\scriptstyle\bullet}\tilde{x}_1\tilde{x}_2\tilde{x}_3\cdots$, where $\tilde{x}_i = x_i$ for $i \neq k$, and $\tilde{x}_k = x_k \pm 1$.

Denote

$$s_i := \underbrace{{\scriptstyle\bullet}y_{i+1}y_{i+2}y_{i+3}\cdots}_{\in[0,1)} - \underbrace{{\scriptstyle\bullet}\tilde{x}_{i+1}\tilde{x}_{i+2}\tilde{x}_{i+3}\cdots}_{\in[-\frac{1}{\beta},1+\frac{1}{\beta})},$$

then $s_i \in (-1-\frac{1}{\beta}, 1+\frac{1}{\beta})$, and we have that $s_{i+1} = \beta s_i + (\tilde{x}_i - y_i)$; we will denote this relation by an arrow $s_i \xrightarrow{\tilde{x}_i - y_i} s_{i+1}$.

Consider $i \leq k-1$. Then the only possible values of s_i and possible arrows are for $i \leq k-2$:

$$0 \xrightarrow{0} 0, \qquad 0 \xrightarrow{\pm 1} \pm{\scriptstyle\bullet}1^d, \qquad \mp{\scriptstyle\bullet}1^n \xrightarrow{\pm 1} \mp{\scriptstyle\bullet}1^{n-1} \quad (n \neq 0).$$

For $i = k-1$, we have additionally:

$$\mp{\scriptstyle\bullet}1^q \xrightarrow{\pm 2} \pm{\scriptstyle\bullet}0^{q-1}1^{d-q+1} \quad (n \neq 0).$$

For $i \geq k$, the arrows change completely since ${\scriptstyle\bullet}\tilde{x}_{i+1}\tilde{x}_{i+2}\cdots \in [0,1)$, whence $s_i \in (-1,1)$. Also, the new states $\pm{\scriptstyle\bullet}0^q1^r$ have to be considered. We get:

$$\pm{\scriptstyle\bullet}0^q1^r \xrightarrow{0} \pm{\scriptstyle\bullet}0^{q-1}1^r \qquad (q \neq 0),$$

$$\mp{\scriptstyle\bullet}0^q1^r \xrightarrow{\pm 1} \pm{\scriptstyle\bullet}1^{q-1}0^r1^{d-q-r+1} \quad (q, r \neq 0),$$

$$\mp{\scriptstyle\bullet}1^q0^r1^t \xrightarrow{\pm 1} \mp{\scriptstyle\bullet}1^{q-1}0^r1^t \qquad (q \neq 0).$$

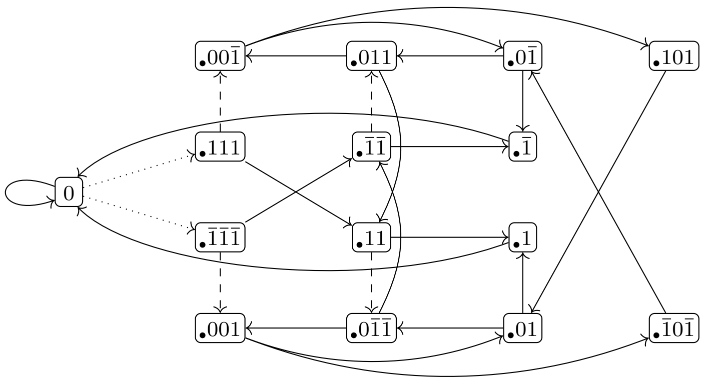

FIGURE 3. The "automaton" built in the proof of Lemma 3, for $d = 3$. The solid arrows represent arrows labelled by 0 or ± 1, the dashed arrows by ± 2 (these are available only for $i = k-1$). The dotted arrows are available only for $i \le k-1$ since they lead to $s_{i+1} = \pm 1$.

Since every d-tuple $\{s_i, s_{i+1}, \dots, s_{i+d-1}\}$ contains a positive element, we can find $i \ge n$ such that $s_i = {\bullet}0^q 1^r \ge 0$ and $\tilde{x}_{i+1}\tilde{x}_{i+2}\tilde{x}_{i+3}\cdots$ is purely periodic, i.e., $\tilde{x}_{i+1}\tilde{x}_{i+2}\tilde{x}_{i+3}\cdots = (p_1 p_2 \dots p_d)^\omega$ for some $p_j \in \{0,1\}$.

There are two cases. First, suppose $p_{q+1}p_{q+2}\cdots p_{q+r} = 0^r$. Then

$$y_{i+1}y_{i+2}y_{i+3}\cdots = p_1 p_2 \dots p_q 1^r p_{q+r+1}\cdots p_d (p_1 \cdots p_d)^\omega.$$

Second, suppose $p_{q+1}p_{q+2}\cdots p_{q+r} \ne 0^r$. Then we can find unique t, u such that

$$p_t p_{t+1}\cdots p_q = 01^{q-t} \quad \text{and} \quad p_{q+u}p_{q+u+1}\cdots p_{q+r} = 10^{r-u}$$

(if we had $p_1 p_2 \cdots p_q = 1^q$, it would be a contradiction with ${\bullet}y_{i+1}y_{i+2}\cdots = s_i + {\bullet}\tilde{x}_{i+1}\tilde{x}_{i+2}\cdots < 1$). Then the new pre-period and period are

$$\begin{aligned} y_{i+1}y_{i+2}\cdots y_{i+d} &= p_1\cdots p_{t-1}10^{q-t}p_{q+1}\cdots p_{q+u-1}01^{r-u}p_{q+r+1}\cdots p_d, \\ y_{i+d+1}y_{i+d+2}\cdots &= \big(p_1\cdots p_{t-1}1p_{t+1}\cdots\cdots p_{q+u-1}0p_{t+u+1}\cdots\cdots\ p_d\big)^\omega, \end{aligned} \tag{3.2}$$

because this value of the sequence $y_{i+1}y_{i+2}\cdots$ is T_{B}-admissible satisfies that

$$\begin{aligned} &{\bullet}y_{i+1}y_{i+2}\cdots - {\bullet}\tilde{x}_{i+1}\tilde{x}_{i+2}\cdots \\ &\quad = {\bullet}0^{t-1}1\bar{1}^{q-t}0^{u-1}\bar{1}1^{r-u}0^{d-q-r}\big(0^{t-1}10^{q+u-t-1}\bar{1}0^{d-q-u}\big)^\omega \\ &\quad = {\bullet}0^t\bar{1}^{q-t}0^r1^{r-u} + {\bullet}\big(0^{t-1}10^{q+u-t-1}\bar{1}0^{d-q-u}\big)^\omega \\ &\quad = {\bullet}0^t\bar{1}^{q-t}0^r1^{r-u} + {\bullet}0^t1^{q+u-t} = {\bullet}0^q1^r = s_i, \end{aligned}$$

and it is T_{B}-admissible. In either case, the sum of the elements of the period is preserved. □

Example 1. We apply the lemma to an example $d = 3$, $(x)_{\mathsf{B}} = 0111011(010)^\omega$ and $z = \beta^{-7}$. Then $\tilde{x}_1\tilde{x}_2\cdots = 0111012(010)^\omega$ and $y_1 y_2 \cdots = 1000100101(100)^\omega$. The computation is as follows:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$\tilde{x}_i$		0	1	1	1	0	1	2	0	1	0	0	1	0	$\cdots$
y_i		1	0	0	0	1	0	0	1	0	1	1	0	0	$\cdots$
s_i	${\bullet}0$	${\bullet}\bar{1}\bar{1}\bar{1}$	${\bullet}\bar{1}\bar{1}$	${\bullet}\bar{1}$	${\bullet}0$	${\bullet}\bar{1}\bar{1}\bar{1}$	${\bullet}\bar{1}\bar{1}$	${\bullet}011$	${\bullet}00\bar{1}$	${\bullet}101$	${\bullet}01$	${\bullet}0\bar{1}\bar{1}$	${\bullet}001$	${\bullet}01$	$\cdots$

(this computation follows the arrows in Figure 3). For $i = 7$, we have that $s_7 = {\bullet}011 \geq 0$ and $x_8x_9\cdots = (100)^\omega$ is purely periodic. Therefore we have $q = 1$ and $r = 2$ and $p_1p_2p_3 = 010$. We have $p_{q+1}\cdots p_{q+r} = 10 \neq 0^r$; we get $t = 1$ and $u = 1$. From (3.2) we confirm that $y_8y_9\cdots = 101(100)^\omega$.

Lemma 4. *Let $h \in \{1, 2, \ldots, d-1\}$. Then the set $[\![h]\!] \cap X_{\mathsf{S}}$ contains exactly such $x \in \mathbb{Q}(\beta) \cap X_{\mathsf{S}}$ that the balanced expansion of $\frac{|x|}{\beta-1}$ has the form*

$$\Big(\frac{|x|}{\beta-1}\Big)_{\mathsf{B}} = x_1x_2\cdots x_n(x_{n+1}x_{n+2}\cdots x_{n+d})^\omega \tag{3.3}$$

$$\text{with} \quad x_{n+1}+x_{n+2}+\cdots+x_{n+d} = \begin{cases} h & \text{if } x>0,\\ d-1 & \text{if } x<0 \text{ and } h=d-1,\\ d-1-h & \text{if } x<0 \text{ and } 1\leq h\leq d-2.\end{cases}$$

Proof. We start by proving that whatever $x \in [\![h]\!] \cap X_{\mathsf{S}}$ we take, it satisfies (3.3). As ${\bullet}1^j \in [\![j]\!]$ for all $j \in \mathbb{N}$, there exists $y \in \mathbb{Z}[\beta]$ such that

$$x = \begin{cases} \frac{y}{\beta-1} + {\bullet}1^h & \text{if } x>0,\\ -\big(\frac{y}{\beta-1} + {\bullet}1^{d-1}\big) & \text{if } x<0 \text{ and } h=d-1,\\ -\big(\frac{y}{\beta-1} + {\bullet}1^{d-1-h}\big) & \text{if } x<0 \text{ and } 1\leq h\leq d-2.\end{cases}$$

Since $(\frac{1}{\beta-1} \times {\bullet}1^j)_{\mathsf{B}} = (1^j0^{d-j})^\omega$, the result follows from Lemma 3.

We finish by proving other direction. Suppose x satisfies (3.3). Without the loss of generality, suppose that the length of the pre-period is a multiple of d, and put $y := (\beta-1)\times {\bullet}(x_{n+1}x_{n+2}\cdots x_{n+d})^\omega = {\bullet}x_{n+1}x_{n+2}\cdots x_{n+d} \in [\![h]\!]$. Then

$$\frac{x-y}{\beta-1} = {\bullet}(x_1{-}x_{n+1})\cdots(x_d{-}x_{n+d})(x_{d+1}{-}x_{n+1})\cdots(x_n{-}x_{n+d})0^\omega \in \mathbb{Z}[\beta].$$

Therefore $x \in [\![y]\!] = [\![h]\!]$. □

Proof of Theorem 3. Let $x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}$. By Lemmas 2 and 4, the symmetric expansion $(|x|)_{\mathsf{S}}$ is periodic with period d. Suppose it is purely periodic. Then by Lemma 2, $(\frac{|x|}{\beta-1})_{\mathsf{B}}$ is also purely periodic; we denote it $(\frac{|x|}{\beta-1})_{\mathsf{B}} = (p_1p_2\cdots p_d)^\omega$. Therefore, since $\frac{1}{\beta-1} = {\bullet}(0^{d-1}1)^\omega$, we have that $|x| = {\bullet}p_1p_2\cdots p_d$. The fact that $p_1 = 0$ follows from $|x| \leq \frac{1}{2} < \frac{1}{\beta}$.

On the other hand, any $x = \pm{\bullet}0p_2\cdots p_d \neq 0$ satisfies that $x \in X_{\mathsf{S}} \cap \mathbb{Z}[\beta]$ and $(\frac{|x|}{\beta-1})_{\mathsf{B}} = (0p_2\cdots p_d)^\omega$ is purely periodic, therefore $x \in \mathcal{P}$. □

Lemma 5. *There exists a number $z \in \mathbb{Z}[\beta]$ such that $\Phi(z)$ lies exactly in $d-1$ tiles.*

Before we prove this lemma, let us recall a helpful result by C. Kalle and W. Steiner:

Lemma 6. [KS12, Proposition 4.15] *Suppose $z \in \mathbb{Z}[\beta] \cap [0,\infty)$. Let $k \in \mathbb{N}$ be an integer such that for all $y \in \mathcal{P}$, the expansions $(y)_{\mathsf{S}}$ and $(y+\beta^{-k}z)_{\mathsf{S}}$ have a common prefix at least as long as the period of y.*

Then $\Phi(z)$ lies in a tile $\mathcal{R}(x)$ for $x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}}$ if and only if

$$x = T_{\mathsf{S}}^k(y+\beta^{-k}z) \quad \textit{for some } y \in \mathcal{P}.$$

Proof of Lemma 5. We put $z := (0^{d-1}1)^{d-1}{\bullet} \in \mathbb{Z}[\beta]\cap[0,\infty)$. Let us fix $y = \pm{\bullet}0y_2y_3\cdots y_d \in \mathcal{P}$. Then we can write y as $y = (-p_1){\bullet}p_1p_2p_3\cdots p_d$, where

$$p_i = y_i \quad \text{if } y \geq 0; \qquad p_i = 1-y_i \quad \text{if } y<0.$$

Note that $h \coloneqq p_1 + p_2 + \cdots + p_d \in \{1, \dots, d-1\}$. Let

$$t \coloneqq \psi\big(y + \beta^{-d^2} z\big) = \frac{1}{\beta - 1} \times {}_\bullet p_1 p_2 \cdots p_d \underbrace{(0^{d-1}1)(0^{d-1}1)\cdots(0^{d-1}1)}_{d-1 \text{ times}}.$$

Defining $f(x) \coloneqq \beta^d x + \frac{1}{\beta-1}$, we get that

$$\beta^{d^2} t = f^{d-1}\Big(\frac{1}{\beta-1} \times p_1 p_2 \cdots p_{d\bullet}\Big) = f^{d-1}\big(p_1 \cdots p_{d\bullet}(p_1 \cdots p_d)^\omega\big),$$

where $f^{d-1}(x)$ denotes the $(d-1)$th iteration $f(f(\cdots f(x)\cdots))$.

We have the following relations:

$$\begin{aligned} &f\big(x_{-N}\cdots x_{0\bullet}(x_1\cdots x_{n-1}01^{d-n})^\omega\big) = x_{-N}\cdots x_{n-1}10^{d-n}{}_\bullet(x_1\cdots x_{n-1}1^{d-n+1})^\omega \\ &\hspace{20em}(\text{if } x_1\cdots x_{n-1} \neq 1^{n-1}); \\ &f\big(x_{-N}\cdots x_{0\bullet}(1^{n-1}01^{d-n})^\omega\big) = x_{-N}\cdots x_0 1^n 0^{d-n-1}1_\bullet(0^{d-1}1)^\omega \\ &\hspace{20em}(\text{if } 1 \le n \le d-1); \\ &f\big(1^{d-1}0_\bullet(1^{d-1}0)^\omega\big) = 1^d 0^{d-1} 1_\bullet (0^{d-1}1)^\omega. \end{aligned} \tag{3.4}$$

It follows that

$$f^{d-h}\big(p_1\cdots p_{d\bullet}(p_1\cdots p_d)^\omega\big) = (\text{something})_\bullet(0^{d-1}1)^\omega,$$
$$\beta^{d^2} t = f^{d-1}\big(p_1\cdots p_{d\bullet}(p_1\cdots p_d)^\omega\big) = t_1 t_2 \cdots t_{d^2\bullet}(0^{d-h}1^h)^\omega.$$

Since the right-hand sides of (3.4) contain neither 0^{d+1} nor 1^{d+1} as a factor, this sequence is T_{B}-admissible, therefore $(t)_{\mathsf{B}} = (\psi(y + \beta^{-d^2}z))_{\mathsf{B}} = t_1 t_2 \cdots t_{d^2}(0^{d-h}1^h)^\omega$.

By Lemma 6, $\Phi(z)$ lies in the tile $\mathcal{R}(x)$ for

$$x = T_{\mathsf{S}}^{d^2}(y + \beta^{-d^2} z) = \psi^{-1} T_{\mathsf{B}}^{d^2}(t).$$

Since $(T_{\mathsf{B}}^{d^2} t)_{\mathsf{B}} = (0^{d-h}1^h)^\omega$, Lemma 2 gives that $(x)_{\mathsf{S}} = (0^{d-h-1}10^{h-1}\bar{1})^\omega$.

Finally, considering all $y \in \mathcal{P}$ at once, we conclude that $\Phi(z)$ lies exactly in tiles $\mathcal{R}({}_\bullet(0^{d-h-1}10^{h-1}\bar{1})^\omega)$ for $h \in \{1, 2, \dots, d-1\}$. That makes $d-1$ tiles. □

Example 2. For $d = 3$, there are 6 purely periodic points $y \in \mathcal{P}$. Following the construction of t in the previous proof we get the following (values of x are the tiles in which $\Phi(z) = 1 + \Phi(\beta^3)$ lies):

y	t	x such that $\Phi(z) \in \mathcal{R}(x)$
${}_\bullet 001$	${}_\bullet 001010101(001)^\omega$	${}_\bullet 001 = {}_\bullet(01\bar{1})^\omega$
${}_\bullet 010$	${}_\bullet 010011101(001)^\omega$	${}_\bullet 001$
${}_\bullet 011$	${}_\bullet 011101010(011)^\omega$	${}_\bullet 011 = {}_\bullet(10\bar{1})^\omega$
${}_\bullet 00\bar{1}$	${}_\bullet 111001010(011)^\omega$	${}_\bullet 011$
${}_\bullet 0\bar{1}0$	${}_\bullet 110001010(011)^\omega$	${}_\bullet 011$
${}_\bullet 0\bar{1}\bar{1}$	${}_\bullet 100110001(001)^\omega$	${}_\bullet 001$

This is in accordance with the previous lemma and also with Figure 4, where $\Phi(z)$ is shown and really lies in $\mathcal{R}({}_\bullet(01\bar{1})^\omega)$ and $\mathcal{R}({}_\bullet(10\bar{1})^\omega)$.

For $d = 4$, we depict a cut through the multiple tiling in Figure 5.

Lemma 7. *For each point $z \in \mathbb{Z}[\beta]$ and for each $h \in \{1, 2, \dots, d-1\}$ there exists $x \in \mathcal{L}_h$ such that $\Phi(z) \in \mathcal{R}(x)$, where $\mathcal{L}_h$ is given by* (1.1).

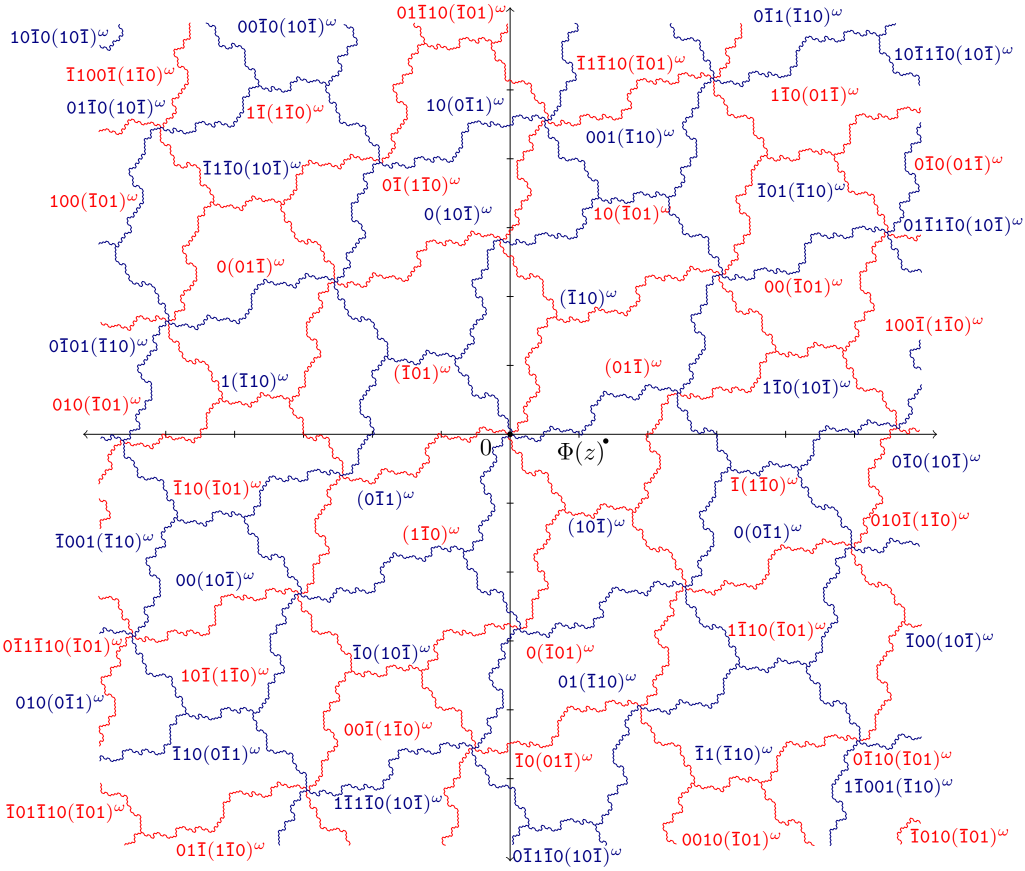

FIGURE 4. The double tiling for the case $d = 3$. The layer $\mathcal{L}_1$ is depicted in red and $\mathcal{L}_2$ in blue. We see that $\Phi(z) = 1 + \Phi(\beta^3) \in \mathcal{R}(\bullet(10\bar{1})^\omega) \cap \mathcal{R}(\bullet(01\bar{1})^\omega)$.

Proof. Suppose $z \geq 0$. Let $k \in \mathbb{N}$ satisfy the hypothesis of Lemma 6. Let $y := \bullet 01^j \in \mathcal{P}$, with $j \in \{1, \ldots, d-1\}$ such that $y + \beta^{-k} z \in [\![h]\!]$. Denote $(\beta^k y + z)_{\mathsf{S}} = x_1 x_2 \cdots$. Then $\Phi(z)$ lies in $\mathcal{R}(x)$ for $x := \bullet x_{k+1} x_{k+2} \cdots$, and $x \in [\![h - \bullet x_0 x_1 \cdots x_k]\!]$. Since $y + \beta^{-k} z > 0$ and the digits 1 and $\bar{1}$ are alternating in $(\beta^k y + z)_{\mathsf{S}}$, we have that

$$[\![\bullet x_0 x_1 \cdots x_k]\!] = [\![x_0 + x_1 + \cdots + x_k]\!] = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x < 0, \end{cases}$$

which means that $x \in \mathcal{L}_h$.

If $z < 0$, we already know that there exists $-x \in \mathcal{L}_{d-h}$ such that $\Phi(-z) \in \mathcal{R}(-x)$, hence $\Phi(z) \in \mathcal{R}(x)$. Since $-[\![h]\!] = [\![d - 1 - h]\!]$, we get that $\mathcal{L}_{d-h} = -\mathcal{L}_h$, therefore $x \in \mathcal{L}_h$. □

Proof of Theorem 1. The collection of tiles $\mathcal{T} = \{\, \mathcal{R}(x) : x \in \mathbb{Z}[\beta] \cap X_{\mathsf{S}} \,\}$ is a multiple tiling by Theorem 4.10 of [KS12]. By Lemma 7, the degree is at least $d-1$ since all points of $\Phi(\mathbb{Z}[\beta])$ lie in at least that many tiles. By Lemma 5, the degree is at most $d-1$ since there exists a point that lies in only $d-1$ tiles. □

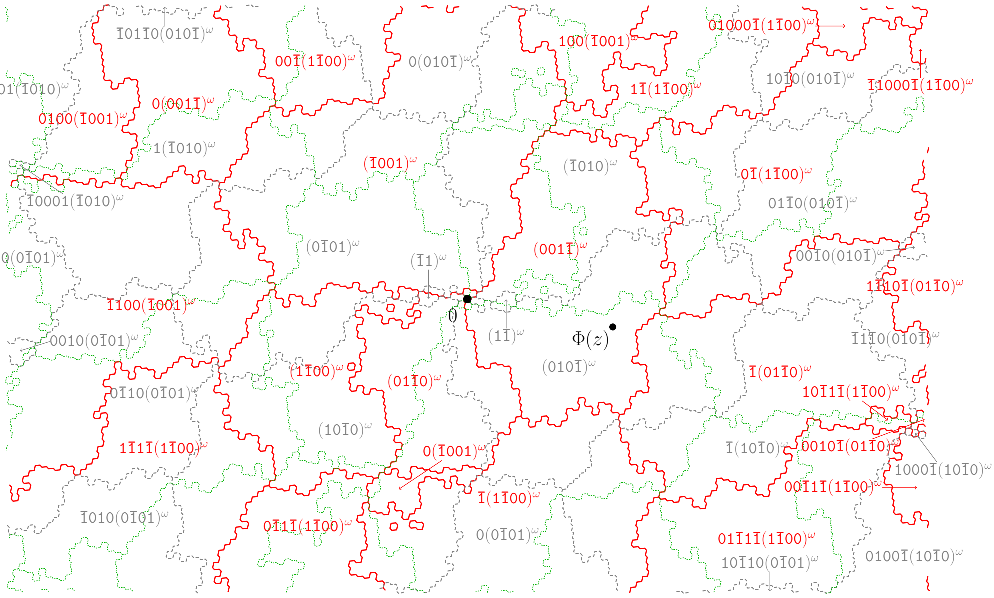

FIGURE 5. A cut through the triple tiling for $d = 4$ that contains the point $\Phi(z) = 1 + \Phi(\beta^4) + \Phi(\beta^8)$. Each layer is depicted in different style and colour: $\mathcal{L}_1$ in solid red, $\mathcal{L}_2$ in dashed gray, and $\mathcal{L}_3$ in dotted green. Since $\mathcal{L}_3 = -\mathcal{L}_1$, the labels for $\mathcal{L}_3$ are omitted.

Proof of Theorem 2. By Lemma 7, each $\Phi(z)$ for $z \in \mathbb{Z}[\beta]$ lies in at least one tile $\mathcal{R}(x)$, $x \in \mathcal{L}_h$, therefore (since $\Phi(\mathbb{Z}[\beta])$ is dense in $\mathbb{R}^{d-1}$ and $\mathcal{R}(x)$ is a closure of its interior) $\bigcup_{x\in\mathcal{L}_h} \mathcal{R}(x) = \mathbb{R}^{d-1}$. Suppose there exists $M \subset \mathbb{R}^{d-1}$ of positive measure such that all $x \in M$ lie in at least two tiles of $\mathcal{L}_h$. These points lie in another $d-2$ tiles, one for each $\tilde{h} \in \{1, 2, \ldots, d-1\} \setminus \{h\}$. Therefore the points of M are covered by d tiles, which is a contradiction with Theorem 1. $\square$

4. Open Problems

Problem 1. Take a (d, a)-Bonacci number for $d \geq 2$ and $a \geq 2$, i.e., the Pisot number $\beta \in (a, a+1)$ satisfying $\beta^d = a\beta^{d-1} + \cdots + a\beta + a$. What is the number of layers of the multiple tiling for the symmetric β-transformation in this case?

Problem 2. Consider the d-Bonacci number β, and the transformation $T_{\beta,l} \colon [l, l+1)$, $x \mapsto \beta x - \lfloor \beta x - l \rfloor$. We know that $T_{\beta,0}$ induces a tiling, since it satisfies Property (F) [FS92]. We prove here that $T_{\beta,-1/2}$ induces a multiple tiling with covering degree $d-1$. What happens if $-\frac{1}{2} < l < 0$? What are the possible values of the covering degree?

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LIAFA, CNRS UMR 7089, Université Paris Diderot – Paris 7, Case 7014, 75205 Paris Cedex 13, France

Dept. Math. FNSPE, Czech Technical University in Prague, Trojanova 13, Prague 12000, Czech Rep.

E-mail address: `tohecz@gmail.com`