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Dissertation study

Representations of real and complex numbers in non-standard numeration systems

Reprezentace reálních a komplexních čísel v nestandardních numeračních systémech

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Contents

1. Introduction

The purpose of this dissertation study is to present the research progress of the student. It comprises the results of paper [**HP**] that is accepted for publication and paper [**HS**] that is in preparation, its extended abstract will appear in the proceedings of Journées Montoises 2014. Both papers are collaborative work of the student with one of his supervisors. The paper [**HP**] treats minimal distances of points in spectra of certain complex Pisot units; it is the first result of this kind for complex numeration systems. The paper [**HS**] studies purely periodic Rényi expansions for the quadratic Pisot non-unit bases.

The study is divided into two main parts. In § 2, first the known results on spectra of real and complex numbers are summarized, and then the results of the paper [**HP**] are presented; last but not least, some ideas for continuation of the work on this topic are given. Similarly organized is § 3, which treats purely periodic expansions. Numerous remarks and comments are in § 4 and the study concludes in § 5. Attached are the most recent versions of the two papers [**HP**, **HS**].

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2. Spectra of Pisot and complex Pisot numbers

2.1. State of the art. For a number $\beta > 1$, its spectrum is a set

$$X^m(\beta) := \{a_0 + a_1\beta + \cdots + a_k\beta^k : k \in \mathbb{N}, a_i \in \{0,1,\ldots,m\}\}, \tag{2.1}$$

where $m \geq 1$ is an integer. The main interest is in determining the minimal distance between two points of the spectrum. If the minimum (more precisely, infimum) of the distances is positive, we say that the spectrum is uniformly discrete, and we denote the minimal distance $\ell_m(\beta)$; otherwise we put $\ell_m(\beta) := 0$. The interest in this problem was initiated by Erdős, Joó, Joó and Komornik [EJK90, EJJ92, EJK98, EK98]. Their main result is in showing the following for the base $\beta > 1$:

Theorem 2.1. (1) *If $m \geq \beta - 1$, then $\ell_m(\beta) \leq 1$;*
(2) *If $m < \beta - 1$, then the set $X^m(\beta)$ is not relatively dense in $[0,\infty)$, i.e., there exist sub-intervals $I \subset [0,\infty)$ of arbitrary length such that $I \cap X^m(\beta) = \emptyset$.*
(3) *If β is a Pisot number, then for all $m \in \mathbb{N}$ we have that $\ell_m(\beta) > 0$, i.e., that $X^m(\beta)$ is uniformly discrete.*

The last statement was later improved by Bugeaud [Bug96], who showed that β is a Pisot number if and only if there exists m_0 such that $X^m(\beta)$ is uniformly discrete for all $m \geq m_0$. Feng [Fen13] further improved it by showing that we can take $m_0 = \lceil \beta \rceil$.

Many authors contributed to determining the value of $\ell_m(\beta)$ for various Pisot numbers β. We would like to mention Komornik, Loreti and Pedicini [KLP00] and Takao Komatsu [Kom02], who provided the results for quadratic Pisot units and all values of $m > \beta - 1$. Feng and Wen [FW02] and Borwein and Hare [BH02] independently of each other took the algorithmic approach; using their results we can determine the value of $\ell_m(\beta)$ for any m and any Pisot number β. Last but not least, Zaïmi [Zaï04] investigated the complementary question: For a fixed $m \in \mathbb{N}$, find $\beta \in (m, m+1)$ such that $\ell_m(\beta)$ is the maximal possible. He showed that $\beta = \frac{m+\sqrt{m^2+4m}}{2}$ satisfies this.

Besides that, Zaïmi started to study the analogous question for complex bases. He was interested in the question for which pairs (γ, m) the set $X^m(\gamma) \subset \mathbb{C}$ is uniformly discrete and he proved a condition similar to the one of Bugeaud: For a fixed $\gamma \in \mathbb{C} \setminus \mathbb{R}$ with $|\gamma| > 1$, the set $X^m(\gamma)$ is uniformly discrete for all $m \in \mathbb{N}$ if and only if γ is a complex Pisot number. Complex Pisot numbers are such algebraic integers that they are non-real, > 1 in modulus, and such that all their Galois conjugates except their complex conjugate are < 1 in modulus.

2.2. **Our contribution.** We now state the main results of paper [**HP**]. The first result generalizes the second statement of Theorem 2.1 to the complex case:

Theorem A. *Let $\gamma \in \mathbb{C} \setminus \mathbb{R}$, $|\gamma| > 1$ be a complex number, let $m \in \mathbb{N}$ satisfy $m < |\gamma|^2 - 1$. Then $X^m(\gamma)$ is not relatively dense in $\mathbb{C}$, i.e., there exist arbitrarily large balls $B = B(x, R) \subset \mathbb{C}$ such that $B \cap X^m(\gamma) = \emptyset$.*

For cubic complex Pisot units, i.e., complex cubic numbers γ with $|\gamma| > 1$ such that their real conjugate γ' satisfies $|\gamma'| < 1$ and $|\gamma|^2 = 1/\gamma'$ we derive the following result.

Theorem B. *Let γ be a cubic complex Pisot unit such that $1/\gamma' > 1$ satisfies Property (F). Then for all $m > |\gamma|^2 - 1$, we have that $X^m(\gamma)$ is a cut-and-project (model) set. There exists an algorithm that for a given base γ computes the value of $\ell_m(\gamma)$ for all m at once.*

We implement the algorithm in Sage [Sage14]. Applying the algorithm to the base complex Tribonacci constant $\gamma_T \approx -0.771 + 1.115\mathrm{i}$, a root of the polynomial $Y^3 + Y^2 + Y - 1$, we obtain the following result:

Theorem C. *Let γ be the complex Tribonacci constant and $m \in \mathbb{N}$. Let $k \in \mathbb{Z}$ be the maximal integer such that $m \geq (1 - \gamma')\left(\frac{1}{\gamma'}\right)^k$, where γ' is the real Galois conjugate of γ. Then we have*

$$\ell_m(\gamma) = |\gamma|^{-k}. \tag{2.2}$$

2.3. **Continuation of the work.** There are several directions in which the research may be continued:

(1) We observed that for $\gamma = \gamma_T$ the complex Tribonacci constant, the values of $\ell_m(\gamma)$ are units. The program in Sage allows us to see that this property is generic and holds for a lot of bases. Therefore we would like to show that it is true for all bases that satisfy the hypothesis of Theorem B.

(2) We treat only bases such that γ' is positive. When considering the bases with $\gamma' \in (-1, 0)$, we need to take into account Property $(-\mathrm{F})$ related to Ito-Sadahiro $(-\beta)$-expansions.

(3) Property (F) is not necessary for our result. It suffices to show that all numbers from $\mathbb{Z}[\gamma']$ can be represented as a finite sum

$$a_k(\gamma')^k + a_{k-1}(\gamma')^{k-1} + \cdots + a_{l+1}(\gamma')^{l+1} + a_l(\gamma')^l,$$

where the digits are taken arbitrarily from the set $a_j \in \{0, \dots, m\}$. We already know from Schmidt [Sch80] that all numbers from $\mathbb{Z}[\gamma']$ have periodic expansions, therefore only such expansions need to be considered. For a lot of bases γ, we were able to find m_0 such that this property is established for all $m > m_0$. However, it is not even clear whether such m_0 exists for all cubic complex Pisot units with $\gamma' > 0$. For instance when γ

is a non-real root of $Y^3 - 4Y^2 + 5Y - 1$, we do not know any value of m that would satisfy this.

(4) The second paper [**HS**] shows that some results on numeration systems with a base that is a unit can be generalized to non-units by considering finite (p-adic) places of the corresponding algebraic field. We know that for γ complex Pisot non-unit, the set $X^m(\gamma)$ is contained in certain cut-and-project sets. The cut-and-project scheme allows more general acceptance windows than subsets of $\mathbb{R}^n$. It should be explored whether $X^m(\gamma)$ for non-unit γ is a cut-and-project set with an unusual acceptance window. This would allow the results to be generalized to a wider class of bases.

3. Purely periodic Rényi expansions

3.1. **State of the art.** Rényi β-expansions [Rén57] provide a very natural generalization of standard positional numeration systems such as the decimal system. Expansions of numbers $x \in [0,1)$ can be defined in terms of a transformation. Let $\beta > 1$ denote the base. Then the Rényi transformation is the map

$$T : [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor. \tag{3.1}$$

The expansion of x is the infinite string $\bullet x_1 x_2 x_3 \cdots$ where $x_j := \lfloor \beta T^{j-1} x \rfloor$. It is a well-known fact that for $b \in \mathbb{N}$, the b-expansion of $x \in [0,1)$ is eventually periodic (i.e., there exists p, n such that $x_{k+p} = x_p$ for all $k \geq n$) if and only if $x \in \mathbb{Q}$. This result was generalized to all Pisot bases by Schmidt [Sch80], who proved that for a Pisot number β the expansion of $x \in [0,1)$ is eventually periodic if and only if x is an element of the algebraic field $\mathbb{Q}(\beta)$. Moreover, he showed that when β satisfies $\beta^2 = a\beta + 1$ with $a \geq 1$, then all $x \in [0,1) \cap \mathbb{Q}$ have a purely periodic β-expansion.

Akiyama [Aki98] showed that if β is a Pisot unit satisfying certain finiteness property called Property (F′) then there exists $c > 0$ such that all rational numbers $x \in \mathbb{Q} \cap [0,c)$ have a purely periodic expansion. If β is not a unit, then only numbers from

$$\mathbb{Q}_{N(\beta)} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(q, N(\beta)) = 1\}$$

can have a purely periodic expansion: All $x \in \mathbb{Q} \setminus \mathbb{Q}_{N(\beta)}$ have eventually periodic expansions. Many Pisot non-units satisfy that there exists $c > 0$ such that all $x \in \mathbb{Q}_{N(\beta)} \cap [0,c)$ have purely periodic expansion. The supremum of c satisfying this property is commonly denoted $\gamma(\beta)$, and we put $\gamma(\beta) = 0$ if no such c exists.

The transformation T possesses an ergodic invariant measure (this was proved already by Rényi in his original paper [Rén57]). Therefore this transformation on the interval $[0,1)$ forms a dynamical system. It is easy to observe that the expansion of x is purely periodic if and only if x is a periodic point of T, i.e., there exists $p \geq 1$ such that $T^p x = x$.

The main tool in the proofs are dynamical properies of the transformation T. The natural extension $(\mathcal{X}, \mathcal{T})$ of $([0,1), T)$ can be defined in an algebraic way. Taking this form of the natural extension, several authors contributed to proving the following result: A point $x \in [0,1)$ has purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and its diagonal embedding lies in the natural extension domain $\mathcal{X}$. The quadratic unit case was solved by Hara and Ito [HI89], the confluent unit case by Ito and Sano [IS01, IS02]. Then Ito and Rao [IR05] resolved the unit case completely using an algebraic argument. For non-unit bases β, one has to consider finite (p-adic) places of the field $\mathbb{Q}(\beta)$. This consideration allowed Berthé and Siegel [BS07] to expand the result to all (non-unit) Pisot numbers.

Recently, Minervino and Steiner [MS14] have described the boundary of $\mathcal{X}$ for quadratic non-unit Pisot bases. This allowed them to find the value of $\gamma(\beta)$:

Theorem 3.1 ([ABBS08, MS14]). *Let β be the positive root of $\beta^2 = a\beta + b$ for $a \geq b > 0$ two co-prime integers. Then*

$$\gamma(\beta) = \begin{cases} 1 - \frac{(b-1)b\beta}{\beta^2 - b^2} \in (0,1) & \text{if } a > b(b-1), \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **Our contribution.** The result of Minervino and Steiner relies on the fact that when a and b are co-prime, the rational numbers are dense in the finite places that need to be considered. This is not the case when a and b have a common divisor. However, we manage to extend this result to such bases:

Theorem D. *Let β be the positive root of $\beta^2 = a\beta + b$ for $a \geq b > 0$ two integers. Then the value of $\gamma(\beta)$ can be computed with arbitrary precision.*

If moreover, a is a multiple of b, then $\gamma(\beta) = 1$ if and only if $a \geq b^2$ or $(a,b) = (24,6), (30,6)$.

It seems that if a, b are not co-prime then $\gamma(\beta)$ is either 0, 1, or does not belong to $\mathbb{Q}(\beta)$. We were able to show when $\gamma(\beta) = 1$ for the case $b \mid a$ because the distribution of rational numbers in the p-adic places is predictable. In the general quadratic case, the behavior of the rational numbers is even more complicated.

3.3. **Continuation of the work.** The limitation of result to quadratic Pisot numbers is given by the fact that only for them, the shape of the natural extension is known. Obtaining the shape of the natural extension for instance for cubic Pisot numbers (in terms of description of the boundary) would allow our approach to be used to determine the value of $\gamma(\beta)$ for these β.

For quadratic β, two questions should be answered:

(1) When is $\gamma(\beta)$ equal to 0 or 1?
(2) When is $\gamma(\beta)$ algebraic and when it is transcendental?

We started to investigate the first question for the case $b \mid a$ and we know when $\gamma(\beta) = 1$. But for showing when $\gamma(\beta) = 0$ we need to consider both boundaries of $\mathcal{X}$; so far only considering the right boundary was sufficient. However, knowledge of the distribution of rational numbers in the p-adic spaces is even more crucial for that.

4. Other results, remarks and comments

4.1. **Balances of $\mathcal{S}$-adic words.** Besides the results mentioned above, we obtained a result in combinatorics on words, namely on balances of $\mathcal{S}$-adic words. If $\mathcal{S}$ is a set of morphisms on infinite words over a finite alphabet $\mathcal{A}$, then we say that an infinite word $\boldsymbol{u} \in \mathcal{A}^\omega$ is $\mathcal{S}$-adic if there exist a sequence of morphisms $\sigma_0, \sigma_1, \sigma_2, \dots \in \mathcal{S}$ and a sequence of letters $a_0, a_1, a_2, \dots \in \mathcal{A}$ such that

$$\boldsymbol{u} = \lim_{n\to\infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1}(a_n),$$

see [BD13]. The balance number of a word $\boldsymbol{u} \in \mathcal{A}^\omega$ is defined as the maximum

$$\max_{a\in\mathcal{A}} \max_{\substack{v,w \text{ factors of } \boldsymbol{u} \\ |v|=|w|}} \big||w|_a - |v|_a\big|,$$

and it is a measure of discrepancy of the symbolic dynamical system associated to the word $\boldsymbol{u}$. The result can be stated as follows:

Theorem E. *Let $\mathcal{S}$ be a set of morphisms on a fixed alphabet such that all incidence matrices of them are regular. Suppose that for all $n \in \mathbb{N}$, there exists an $\mathcal{S}$-adic word such that its balance number is $> n$. Then there exists an $\mathcal{S}$-adic word whose balance number is $+\infty$.*

The proof of this statement for a specific $\mathcal{S}$-adic system that codes Brun expansions is given in [**DHS**]. Its generalization to all $\mathcal{S}$-adic systems with regular incidence matrices is straightforward. Let us remark that the idea was used before by Berthé, Cassaigne and Steiner [BCS13] for Arnoux-Rauzy words.

4.2. **Multiplication of Rényi expansions.** Let us ask yet another question: How to multiply (eventually) periodic Rényi expansions? To the best of our knowledge, this question has not been treated yet for non-integer bases.

If a reasonably fast algorithm for multiplication was provided, it would allow software such as Sage [Sage14] to perform precise arithmetics in fields $\mathbb{Q}(\beta)$, and at the same time, to be able to compare the values of the numbers fast. This is due to the fact that Rényi expansions preserve the natural order on real numbers. Currently, the numbers are stored in powers of β, which makes the comparison difficult.

We managed to obtain a partial answer to this problem and we construct an algorithm for multiplication of eventually periodic expansions in the base Golden mean. However, the algorithm is currently clumsy, it needs some polishing, and the rigorous proof of its correctness is in preparation. Altogether, this needs to be considered as a work in progress, since basically only ideas were brought up. It is aslo necessary to study for what bases besides the Golden mean our approach works.

4.3. **Connection between the results.** There is one thing that connects all mentioned work on numeration systems: In all the results, periodic expansions play a significant role. In the study of complex spectra, we mention that instead of Property (F), it is sufficient to find arbitrary finite representations for numbers purely periodic Rényi expansions. The other two results on purely periodic expansions of rational numbers and on addition of periodic expansions have the periodicity already in the title.

5. Conclusions

We obtained several results in both complex spectra and in purely periodic Rényi expansions. Besides that, minor results in another related topics were presented. Numerous suggestions for future work were given.

List of articles with the collaboration of the student

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[**HP**] Tomáš Hejda and Edita Pelantová, *Spectral properties of cubic complex Pisot units*, 2014, to appear in Math Comp.

[**HS**] Tomáš Hejda and Wolfgang Steiner, *Purely periodic expansions for the quadratic non-unit bases*, 2014, preprint. Extended abstract will appear in Journées Montoises 2014.

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SPECTRAL PROPERTIES OF CUBIC COMPLEX PISOT UNITS

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ABSTRACT. For a real number $\beta > 1$, Erdős, Joó and Komornik study distances between consecutive points in the set

$$X^m(\beta) = \Big\{ \sum_{j=0}^{n} a_j \beta^j : n \in \mathbb{N},\, a_k \in \{0,1,\dots,m\} \Big\}.$$

Pisot numbers play a crucial role for the properties of $X^m(\beta)$. Following the work of Zaïmi, who considered $X^m(\gamma)$ with $\gamma \in \mathbb{C} \setminus \mathbb{R}$ and $|\gamma| > 1$, we show that for any non-real γ and $m < |\gamma|^2 - 1$, the set $X^m(\gamma)$ is not relatively dense in the complex plane.

Then we focus on complex Pisot units γ with a positive real conjugate γ' and $m > |\gamma|^2 - 1$. If the number $1/\gamma'$ satisfies Property (F), we deduce that $X^m(\gamma)$ is uniformly discrete and relatively dense, i.e., $X^m(\gamma)$ is a Delone set. Moreover, we present an algorithm for determining two parameters of the Delone set $X^m(\gamma)$ which are analogous to minimal and maximal distances in the real case $X^m(\beta)$. For γ satisfying $\gamma^3 + \gamma^2 + \gamma - 1 = 0$, explicit formulas for the two parameters are given.

1. INTRODUCTION

In [EJK90, EJK98], Erdős, Joó and Komornik studied the set

$$X^m(\beta) := \Big\{ \sum_{j=0}^{n} a_j \beta^j : n \in \mathbb{N},\, a_k \in \{0,1,\dots,m\} \Big\},$$

where $\beta > 1$. Since this set has no accumulation points, we can find an increasing sequence

$$0 = x_0 < x_1 < x_2 < \cdots < x_k < \cdots$$

such that $X^m(\beta) = \{x_k : k \in \mathbb{N}\}$. The research of Erdős et al.aims to describe distances between consecutive points of $X^m(\beta)$, i.e., the sequence $(x_{k+1} - x_k)_{k \in \mathbb{N}}$. The properties of this sequence depend on the value $m \in \mathbb{N}$. It is easy to show that when $m \geq \beta - 1$, we have $x_{k+1} - x_k \leq 1$ for all $k \geq 0$; and when $m < \beta - 1$, the distances $x_{k+1} - x_k$ can be arbitrarily large.

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Also, many properties of $X^m(\beta)$ depend on β being a *Pisot number* (i.e., an algebraic integer > 1 such that all its Galois conjugates are < 1 in modulus). Bugeaud [Bug96] showed that

$$\ell_m(\beta) \coloneqq \liminf_{k\to\infty}(x_{k+1}-x_k) > 0 \quad \text{for all } m \in \mathbb{N}$$

if and only if base β is a Pisot number. Recently, Feng [Fen13] proved a stronger result that the bound $\beta-1$ for the alphabet size is crucial. In particular, $\ell_m(\beta)=0$ if and only if $m > \beta-1$ and β is not a Pisot number.

Therefore, the case β Pisot and $m > \beta-1$ has been further studied. From the approximation property of Pisot numbers we know that for a fixed β and $m > \beta-1$ the sequence $(x_{k+1}-x_k)$ takes only finitely many values. Feng and Wen [FW02] used this fact to show that the sequence of distances $(x_{k+1}-x_k)$ is substitutive: roughly speaking, it can be generated by a system of rewriting rules over a finite alphabet. This allows us, for a fixed β and m, to determine values of all distances $(x_{k+1}-x_k)$ and subsequently the value of $\ell_m(\beta)$. An algorithm for obtaining the minimal distance $\ell_m(\beta)$ for certain β was also proposed by Borwein and Hare [BH02].

The first formula which determines the value of $\ell_m(\beta)$ for all m at once appeared in 2000: Komornik, Loreti and Pedicini [KLP00] studied the base Golden mean. The generalization of this result to all quadratic Pisot units was provided by Takao Komatsu [Kom02] in 2002.

To the best of our knowledge, the value of

$$L_m(\beta) \coloneqq \limsup_{k\to\infty}(x_{k+1}-x_k) \tag{1.1}$$

for all m is only known for the base Golden mean, due to Borwein and Hare [BH03]. Of course, for a given m, the value of $L_m(\beta)$ can be computed using [FW02].

Zaïmi [Zaï04] was interested in a complementary question: Fix the alphabet size, i.e., the maximal digit m, and look for the extreme values of $\ell_m(\beta)$, where β runs through Pisot numbers in $(m, m+1)$. He showed that $\ell_m(\beta)$ is maximized for certain quadratic Pisot numbers.

Besides that, Zaïmi started the study of the set $X^m(\gamma)$, where γ is a complex number > 1 in modulus, and he put

$$\ell_m(\gamma) \coloneqq \inf\big\{|x-y| : x, y \in X^m(\gamma),\, x \neq y\big\}. \tag{1.2}$$

He proved an analogous result to the one for real bases by Bugeaud, namely that $\ell_m(\gamma) > 0$ for all m if and only if γ is a complex Pisot number, where a *complex Pisot number* is defined as a non-real algebraic integer > 1 in modulus whose Galois conjugates except its complex conjugate are < 1 in modulus.

In the complex plane, $\ell_m(\gamma)$ and $L_m(\gamma)$ cannot be defined as simply as in the real case since we have no natural ordering of the set $X^m(\gamma)$ in $\mathbb{C}$. To overcome this difficulty, we were inspired by notions used in the definition of Delone sets. We say that a set Σ is:

- *uniformly discrete* if there exists $d > 0$ such that $|x-y| \geq d$ for all distinct $x, y \in \Sigma$;
- *relatively dense* if there exists $D > 0$ such that for all $x \in \mathbb{C}$ the closed ball $B(x, D/2) = \{z \in \mathbb{C} : |z - x| \leq D/2\}$ contains a point from Σ.

A set that is both uniformly discrete and relatively dense is called a *Delone set.*

Clearly, if $\ell_m(\gamma)$ as given by (1.2) is positive, then $X^m(\gamma)$ is uniformly discrete and $\ell_m(\gamma)$ is the maximal d in the definition of uniform discreteness. Hence $X^m(\gamma)$ is uniformly discrete for all m, when γ is a complex Pisot number.

Let us define

$$L_m(\gamma) := \inf\bigl\{D > 0 : B(x, D/2) \cap X^m(\gamma) \neq \emptyset \text{ for all } x \in \mathbb{C}\bigr\}.$$

In particular, $L_m(\gamma) = +\infty$ if and only if $X^m(\gamma)$ is not relatively dense.

The question for which pairs (γ, m) the set $X^m(\gamma)$ is uniformly discrete or is relatively dense is far from being solved. We provide a necessary condition for relative denseness and we show that in certain cases, it is sufficient as well:

Theorem 1.1. *Let $\gamma \in \mathbb{C}$ be a non-real number > 1 in modulus.*

(i) *If $m < |\gamma|^2 - 1$, then $X^m(\gamma)$ is not relatively dense.*
(ii) [Zaï04] *If $m > |\gamma|^2 - 1$ and γ is not an algebraic number, then $X^m(\gamma)$ is not uniformly discrete.*

The aim of this article is to study the sets $X^m(\gamma)$ simultaneously for all $m \in \mathbb{N}$, for a certain class of cubic complex Pisot units with a positive conjugate γ'. For such γ the Rényi expansions in base $\beta := 1/\gamma'$ have nice properties, which will be crucial in the proofs. When this base satisfies a certain finiteness property, called Property (F) in [Aki00], we show that for all sufficiently large m the set $X^m(\gamma) \subseteq \mathbb{C}$ is a cut-and-project set; roughly speaking, $X^m(\gamma)$ is formed by projections of points from the lattice $\mathbb{Z}^3$ which lie in a sector bounded by two parallel planes in $\mathbb{R}^3$; see Theorem 4.1. From that, the asymptotic behaviour of $\ell_m(\gamma)$ and $L_m(\gamma)$ follows easily, namely:

$$\ell_m(\gamma) = \Theta(1/\sqrt{m}) \quad \text{and} \quad L_m(\gamma) = \Theta(1/\sqrt{m}), \tag{1.3}$$

where $f(m) = \Theta(1/\sqrt{m})$ means that $K_1/\sqrt{m} \leq f(m) \leq K_2/\sqrt{m}$ for some positive constants K_1, K_2.

The method of inspection of Voronoi cells for a specific cut-and-project set, as established by Masáková, Patera and Zich [MPZ03a, MPZ03b, MPZ05], enables us to give a general formula for both $\ell_m(\gamma)$ and $L_m(\gamma)$. In the case where $\gamma = \gamma_T \approx -0.771 + 1.115\mathrm{i}$ is the complex Tribonacci constant, i.e., the complex root of $Y^3 + Y^2 + Y - 1$ with a positive imaginary part, we get the following result:

Theorem 1.2. *Let γ be a complex root of the polynomial Y^3+Y^2+Y-1, $m \in \mathbb{N}$, and $k \in \mathbb{Z}$ be the greatest integer such that $m \geq (1-\gamma')\bigl(\frac{1}{\gamma'}\bigr)^k$, where γ' is the real*

Galois conjugate of γ*. Then we have*

$$\ell_m(\gamma) = |\gamma|^{-k} \quad \textit{and} \quad L_m(\gamma) = 2\sqrt{\frac{1-(\gamma')^2}{3-(\gamma')^2}}|\gamma|^{3-k}. \tag{1.4}$$

The article is organized as follows. In Section 2, we recall certain notions from the theory of β-expansions. Section 3 provides the proof of the 1st part of Theorem 1.1. In Section 4 we prove that $X^m(\gamma)$ is a cut-and-project set in certain cases. Section 5 describes the algorithms for computing $\ell_m(\gamma)$ and $L_m(\gamma)$. These algorithms are applied to the complex Tribonacci number in Section 6, providing the proof of Theorem 1.2. In Section 7 we compute another characteristic of $X^m(\gamma)$ that is based on Delone tessellations. Comments and open problems are in Section 8.

All computations were carried out in Sage [Sage]. The pictures were drawn using Ti*k*Z [TikZ].

2. Preliminaries

2.1. **β-numeration.** Let us recall some facts concerning β-expansions. For a real base $\beta > 1$, and for a number $x \geq 0$, there exist a unique $N \in \mathbb{Z}$ and unique integer coefficients $a_N, a_{N-1}, a_{N-2}, \dots$ such that $a_N \neq 0$ and

$$0 \leq x - \sum_{j=n}^{N} a_j \beta^j < \beta^n \quad \text{for all } n \leq N.$$

The string $a_N a_{N-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots$ is then called the *Rényi expansion* of x in base β [Rén57]. We immediately see that $a_j \in \{k \in \mathbb{Z} : 0 \leq k < \beta\}$. For $\beta \notin \mathbb{Z}$, it means that $a_j \in \{0, \dots, \lfloor \beta \rfloor\}$, where $\lfloor \beta \rfloor$ denotes the greatest integer $\leq \beta$. If only finitely many a_j's are non-zero, we speak about the *finite Rényi expansion* of x. The set of numbers $x \in \mathbb{R}$ such that $|x|$ has a finite Rényi expansion is denoted $\mathrm{Fin}(\beta)$. We say that $\beta > 1$ satisfies *Property (F)* if $\mathrm{Fin}(\beta)$ is a ring, i.e., $\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]$, where $\mathbb{Z}[y]$ denotes as usual the integer combinations of powers of y.

2.2. **Complex Pisot numbers.** We widely use the algebraic properties of a cubic complex Pisot number γ. Such a number has two other Galois conjugates. One of them is the complex conjugate $\overline{\gamma}$. The second one is real and < 1 in modulus; we denote it γ'; we have either $-1 < \gamma' < 0$ or $0 < \gamma' < 1$. In general, for $z \in \mathbb{Q}(\gamma)$ we denote by $z' \in \mathbb{Q}(\gamma') \subset \mathbb{R}$ its image under the Galois isomorphism that maps $\gamma \mapsto \gamma'$. When γ is a unit (i.e., the constant term of its minimal polynomial is ± 1), we know that $\mathbb{Z}[1/\gamma] = \mathbb{Z}[\gamma] = \gamma\mathbb{Z}[\gamma]$.

The method we present here can be applied only in the case when:

(2.1) γ is a cubic complex Pisot unit, its real Galois conjugate γ' is positive, and $\beta := 1/\gamma'$ has Property (F).

It implies that the minimal polynomial of γ is of the form Y^3+bY^2+aY-1 with $a,b\in\mathbb{Z}$. Such a polynomial has a complex root if and only if its discriminant is negative, i.e.,

$$-18ab-4a^3+a^2b^2+4b^3-27<0.$$

The number $\beta=1/\gamma'$ is a root of Y^3-aY^2-bY-1. Akiyama [Aki00] showed that β has Property (F) if and only if

$$|b-1|\le a \quad \text{and} \quad b\ge -1.$$

Therefore we are interested in cases where both conditions are satisfied.

In particular, the *complex Tribonacci constant* $\gamma_T\approx -0.771+1.115\mathrm{i}$ (the root of Y^3+Y^2+Y-1 with a positive imaginary part) satisfies (2.1), as well as the complex roots of polynomials Y^3+bY^2+aY-1 for $b=0,\pm1$ and $a\ge 1$, with the exception $(a,b)=(1,-1)$.

3. Proof of Theorem 1.1

We prove the first part of Theorem 1.1. We cannot easily follow the lines of the proof of the result for the real case (i.e., that $m<\beta-1$ implies $L^m(\beta)=+\infty$), because it relies on the natural ordering of $\mathbb{R}$. In the proof of the theorem, the following 'folklore' lemma about the asymptotic density of relatively dense sets is used:

Lemma 3.1. *Let $\Sigma\subset\mathbb{C}$ be a relatively dense set. Then*

$$\liminf_{r\to\infty}\frac{\#\big(\Sigma\cap B(0,r)\big)}{r^2}>0, \tag{3.1}$$

where $\#A$ is the number of elements of the set A.

Proof. Since Σ is relatively dense, there exists $\lambda>0$ such that every square in $\mathbb{C}$ with side λ contains a point of Σ. Therefore every cell of the lattice $\lambda\mathbb{Z}[\mathrm{i}]=\{\lambda a+\mathrm{i}\lambda b : a,b\in\mathbb{Z}\}$ contains a point of Σ. Since $B(0,r)$ contains at least n^2 cells, where $n=\lfloor r\sqrt{2}/\lambda\rfloor$, we get

$$\liminf_{r\to\infty}\frac{\#\big(\Sigma\cap B(0,r)\big)}{r^2}\ge\liminf_{r\to\infty}\frac{\lfloor r\sqrt{2}/\lambda\rfloor^2}{r^2}=\frac{2}{\lambda^2}>0. \qquad \square$$

Proof of Theorem 1.1, 1st statement. For simplicity, we denote $\Sigma:=X^m(\gamma)$.

First, we show that for any $r\ge m$ we have

$$\Sigma\cap B\big(0,|\gamma|r-m\big)\subseteq\gamma\big(\Sigma\cap B(0,r)\big)+\{0,\dots,m\}$$

and therefore

$$\#\big(\Sigma\cap B(0,|\gamma|r-m)\big)\le(m+1)\#\big(\Sigma\cap B(0,r)\big). \tag{3.2}$$

To prove this, consider $x=\sum_{j=0}^k a_j\gamma^j$ with $a_j\in\{0,\dots,m\}$ and such that $|x|\le|\gamma|r-m$. Then $y:=(x-a_0)/\gamma=\sum_{j=1}^k a_j\gamma^{j-1}\in\Sigma$ and $|y|\le(|x|+a_0)/|\gamma|\le(|\gamma|r-m+m)/|\gamma|=r$. Since $x=\gamma y+a_0$, the inclusion is valid.

Our aim is to prove that under the assumption $m < |\gamma|^2 - 1$, the set Σ is not relatively dense. According to Lemma 3.1, it is enough to construct a sequence (r_k) such that $r_k \to \infty$ and

$$\lim_{k\to\infty} n_k = 0, \quad \text{where} \quad n_k := \frac{\#\big(\Sigma \cap B(0,r_k)\big)}{r_k^2}.$$

Since $\Sigma = X^m(\gamma)$ always contains 0, the set $\Sigma \cap B(0,r_k)$ is non-empty and we have that $n_k > 0$.

Consider a sequence given by the recurrence relation $r_{k+1} = |\gamma| r_k - m$ and $r_0 := |\gamma|^2 + \frac{m}{|\gamma|-1} > m$. The choice of r_0 guarantees that $r_k = |\gamma|^{k+2} + \frac{m}{|\gamma|-1}$, therefore $r_k \to \infty$ and $r_{k+1}/r_k \to |\gamma|$. Then (3.2) gives $\#\big(\Sigma \cap B(0,r_{k+1})\big) \leq (m+1)\#\big(\Sigma \cap B(0,r_k)\big)$, which yields

$$\frac{n_{k+1}}{n_k} \leq \frac{(m+1)r_k^2}{r_{k+1}^2} \xrightarrow{k\to\infty} \frac{m+1}{|\gamma|^2} < 1,$$

therefore $n_k \to 0$ as desired. $\square$

4. Cut-and-project sets versus $X^m(\gamma)$

A cut-and-project scheme in dimension $d+e$ consists of two linear maps $\Psi : \mathbb{R}^{d+e} \to \mathbb{R}^d$ and $\Phi : \mathbb{R}^{d+e} \to \mathbb{R}^e$ satisfying:

(1) $\Psi(\mathbb{R}^{d+e}) = \mathbb{R}^d$ and the restriction of Ψ to the lattice $\mathbb{Z}^{d+e}$ is injective;
(2) the set $\Phi(\mathbb{Z}^{d+e})$ is dense in $\mathbb{R}^e$.

Let $\Omega \subset \mathbb{R}^e$ be a nonempty bounded set such that its closure equals the closure of its interior, i.e., $\overline{\Omega} = \overline{\Omega^\circ}$. Then the set

$$\Sigma(\Omega) := \big\{\Psi(v) : v \in \mathbb{Z}^{d+e}, \Phi(v) \in \Omega\big\} \subseteq \mathbb{R}^d$$

is called a *cut-and-project* set with acceptance window Ω. Cut-and-project sets can be defined in a slightly more general way, cf. [Moo97]. The assumptions made on the acceptance window Ω ensure that every cut-and-project set is a Delone set.

We use the concept of cut-and-project sets for $d = 2$ and $e = 1$. With a slight abuse of notation, we consider $\Psi : \mathbb{R}^3 \to \mathbb{C} \simeq \mathbb{R}^2$. Then it is straightforward that for a cubic complex Pisot number γ, the set defined by

$$\Sigma_\gamma(\Omega) = \big\{z \in \mathbb{Z}[\gamma] : z' \in \Omega\big\}, \quad \text{where } \Omega \subseteq \mathbb{R} \text{ is an interval}, \tag{4.1}$$

is a cut-and-project set. Really, we have

$$\Psi_\gamma(v_0,v_1,v_2) = v_0 + v_1\gamma + v_2\gamma^2 \simeq \begin{pmatrix} \Re(v_0+v_1\gamma+v_2\gamma^2) \\ \Im(v_0+v_1\gamma+v_2\gamma^2) \end{pmatrix}$$
$$\text{and} \quad \Phi_\gamma(v_0,v_1,v_2) = v_0 + v_1\gamma' + v_2(\gamma')^2.$$

We will omit the index γ in the sequel. We now show how $X^m(\gamma)$ fit into the cut-and-project scheme:

Theorem 4.1. *Let γ be a cubic complex Pisot unit with a positive conjugate γ', and let m be an integer $m \geq |\gamma|^2 - 1$. Suppose that base $1/\gamma'$ has Property (F). Then $X^m(\gamma)$ is a cut-and-project set, namely*

$$X^m(\gamma) = \Sigma(\Omega) = \big\{z \in \mathbb{Z}[\gamma] : z' \in \Omega\big\} \quad \textit{with } \Omega = \big[0, m/(1-\gamma')\big). \tag{4.2}$$

Proof. Inclusion $\subseteq$: Let $z \in X^m(\gamma)$. Then $z = \sum_{j=0}^{n} a_j \gamma^j$ with $a_j \in \{0, \ldots, m\}$ and clearly $z \in \mathbb{Z}[\gamma]$. Moreover,

$$0 \leq z' = \sum_{j=0}^{n} a_j (\gamma')^j \leq \sum_{j=0}^{n} m(\gamma')^j < \frac{m}{1-\gamma'}.$$

Inclusion $\supseteq$: Let us take $z \in \mathbb{Z}[\gamma]$ with $z' \in \Omega$. Denote $\beta = 1/\gamma' = \gamma\overline{\gamma} = |\gamma|^2$. We discuss the following two cases:

(1) Suppose $0 \leq z' < 1$. The real base β has Property (F) by the hypothesis. Therefore every number from $\mathbb{Z}[1/\beta] \cap [0, 1)$ has a finite expansion $0.a_1a_2a_3 \ldots a_n$ over the alphabet $\{0, \ldots, m_0\}$, where $m_0 := \lfloor \beta \rfloor$ (the expansion certainly starts after the fractional point since $z < 1$). This means that $z' = \sum_{j=1}^{n} a_j \beta^{-j}$ and therefore $z = \sum_{j=1}^{n} a_j \gamma^j \in X^{m_0}(\gamma)$. Since $X^{m_0}(\gamma) \subseteq X^m(\gamma)$, we get $z \in X^m(\gamma)$.

(2) Suppose $1 \leq z' < m/(1-\gamma')$. Since $z' < \sum_{j=0}^{\infty} m\beta^{-j}$, there exists a minimal $k \geq 0$ such that $z' - \sum_{j=0}^{k} m\beta^{-j} < 0$. Let $b \in \{0, \ldots, m\}$ be such that

$$0 \leq z' - \sum_{j=0}^{k-1} m\beta^{-j} - b\beta^{-k} < \beta^{-k},$$

where $\sum_{j=0}^{-1} m\beta^{-j} := 0$. Then

$$u' := \beta^k \left(z' - \sum_{j=0}^{k-1} m\beta^{-j} - b\beta^{-k} \right)$$

satisfies $0 \leq u' < 1$, and by the previous case there exist $a_1, \ldots, a_n \in \{0, \ldots, m_0\}$ such that $u' = \sum_{j=1}^{n} a_j \beta^{-j}$. Altogether,

$$z' = \sum_{j=0}^{k-1} m(\gamma')^j + b(\gamma')^k + \sum_{j=k+1}^{k+n} a_{j-k}(\gamma')^j$$

and $z \in X^m(\gamma)$. □

The property of cut-and-project sets which allows us to determine the values of $\ell_m(\gamma)$ and $L_m(\gamma)$ is the self-similarity. We say that a Delone set $\Sigma \subseteq \mathbb{C}$ is *self-similar* with a factor $\kappa \in \mathbb{C}$, $|\kappa| > 1$, if $\kappa\Sigma \subseteq \Sigma$. In general, cut-and-project sets are not self-similar. In our special case (4.1), not only the sets are self-similar, but we can prove even a stronger property that will be useful later:

Proposition 4.2. *Let γ be a cubic complex Pisot unit. Then*

$$\Sigma\big((\gamma')^k\Omega\big) = \gamma^k\Sigma(\Omega) \quad \textit{for any interval } \Omega \textit{ and any } k \in \mathbb{Z}.$$

In particular, if $\Omega = [0, c)$ and γ' is positive, then $\gamma'\Omega \subseteq \Omega$ and $\gamma\Sigma \subseteq \Sigma$.

Proof. We prove the claim for $k = \pm 1$, the general case follows by induction. Because $\mathbb{Z}[\gamma] = \gamma\mathbb{Z}[\gamma]$, we have that

$$\begin{aligned}(4.3)\quad \Sigma(\gamma'\Omega) = \big\{x \in \gamma\mathbb{Z}[\gamma] : x' \in \gamma'\Omega\big\} &= \big\{x \in \gamma\mathbb{Z}[\gamma] : \tfrac{1}{\gamma'}x' \in \Omega\big\} \\ &= \gamma\big\{y \in \mathbb{Z}[\gamma] : y' \in \Omega\big\} = \gamma\Sigma(\Omega),\end{aligned}$$

which implies the validity of the statement for $k = +1$. If we apply (4.3) to the window $\tilde{\Omega} = \gamma'\Omega$, we get $\Sigma(\tilde{\Omega}) = \gamma\Sigma(\frac{1}{\gamma'}\tilde{\Omega})$, i.e., $\frac{1}{\gamma}\Sigma(\tilde{\Omega}) = \Sigma(\frac{1}{\gamma'}\tilde{\Omega})$, which implies the validity of the statement for $k = -1$. □

Remark 4.3. Theorem 4.1 and Proposition 4.2 imply the asymptotic behaviour of $\ell_m(\gamma)$ and $L_m(\gamma)$ as described in (1.3), because $|\gamma'| = 1/\sqrt{|\gamma|}$.

5. Voronoi tessellations

For a Delone set Σ, the *Voronoi cell* of a point $x \in \Sigma$ is the set of points which are closer to x than to any other point in Σ. Formally

$$(5.1)\qquad \mathcal{T}(x) := \big\{z \in \mathbb{C} : |z - x| \le |z - y| \text{ for all } y \in \Sigma\big\}.$$

The cell is a convex polygon having x as an interior point. Clearly $\bigcup_{x\in\Sigma}\mathcal{T}(x) = \mathbb{C}$ and the interiors of two cells do not intersect. Such a collection of cells $\{\mathcal{T}(x) : x \in \Sigma\}$ is called a *tessellation* of the complex plane. For every cell $\mathcal{T}(x)$ we define two characteristics:

- $\delta(\mathcal{T}(x))$ is the maximal diameter $d > 0$ such that $B(x, d/2) \subseteq \mathcal{T}(x)$;
- $\Delta(\mathcal{T}(x))$ is the minimal diameter $D > 0$ such that $\mathcal{T}(x) \subseteq B(x, D/2)$.

These δ and Δ allow us to compute the values of $\ell_m(\gamma)$ and $L_m(\gamma)$, namely

$$\ell_m(\gamma) = \inf_x \delta\big(\mathcal{T}(x)\big) \quad\text{and}\quad L_m(\gamma) = \sup_x \Delta\big(\mathcal{T}(x)\big),$$

where x runs the whole set $\Sigma = X^m(\gamma)$.

A *protocell* of a point x is the set $\mathcal{T}(x) - x$. We can define δ, Δ analogously for the protocells. The set of all protocells of the tessellation of Σ is called the *palette* of Σ. We therefore obtain that

$$(5.2)\qquad \ell_m(\gamma) = \inf_{\mathcal{T}} \delta(\mathcal{T}) \quad\text{and}\quad L_m(\gamma) = \sup_{\mathcal{T}} \Delta(\mathcal{T}),$$

where $\mathcal{T}$ runs the whole palette of Σ.

For computing $\delta(\mathcal{T})$ and $\Delta(\mathcal{T})$, we modify the approach of [MPZ03a], where 2-dimensional cut-and-project sets based on quadratic irrationalities are concerned. To find the Voronoi cell of a point $x \in \Sigma(\Omega)$ one does not need to

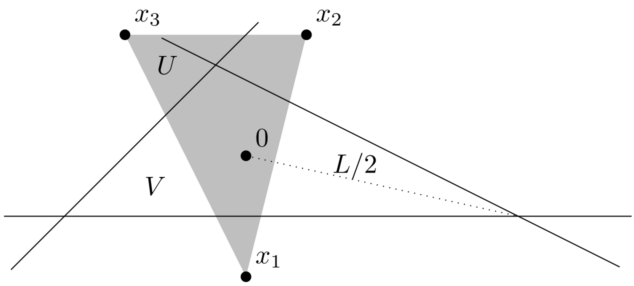


FIGURE 1. To the proof of Lemma 5.1.

consider all points $y \in \Sigma(\Omega)$. It is easy to see that only points y closer to x than $\Delta(\mathcal{T}(x))$ influence the shape of the tile $\mathcal{T}(x)$, i.e.,

$$\mathcal{T}(x) = \big\{z \in \mathbb{C} : |z-x| \le |z-y| \text{ for } y \in \Sigma(\Omega),\ |y-x| \le \Delta(\mathcal{T}(x))\big\}. \tag{5.3}$$

But before the shape of $\mathcal{T}(x)$ is known, we do not know the value of $\Delta(\mathcal{T}(x))$. So we need to find some positive constant L such that

$$\Delta\big(\mathcal{T}(y)\big) \le L \quad \text{for all} \quad y \in \Sigma(\Omega). \tag{5.4}$$

In the rest of this section, we consider cut-and-project sets $\Sigma(\Omega)$ as given by (4.1), where γ satisfies (2.1), i.e., $1/\gamma'$ has Property (F), and where $\Omega = [0, c)$ with $c > 0$ (however, not necessarily of the form $c = \frac{m}{1-\gamma'}$). We denote by $\Re z = \frac{z+\overline{z}}{2}$ and $\Im z = \frac{z-\overline{z}}{2\mathrm{i}}$ respectively the real and the imaginary part of $z \in \mathbb{C}$.

Lemma 5.1. *Let $\Omega = [0, c)$ be an interval. Let p be the first positive integer such that $\Im(\gamma^p)$ and $\Im\gamma$ have the opposite signs and let k be the smallest integer satisfying $(\gamma')^k < c/2$. Then*

$$L := |\gamma|^k \max_{\substack{i,j\in\{0,p-1,p\}\\ i<j}} \left|\frac{\gamma^{i+j}(\gamma^i - \gamma^j)}{\Im\big(\gamma^i\overline{\gamma}^j\big)}\right| \tag{5.5}$$

satisfies $\Delta(\mathcal{T}(y)) \le L$ for all $y \in \Sigma(\Omega)$.

Proof. We first prove the statement for $y = 0$. The choice of k guarantees that $x_1 := \gamma^k$, $x_2 := \gamma^{k+p-1}$ and $x_3 := \gamma^{k+p}$ satisfy $x_1, x_2, x_3 \in \Sigma(\Omega)$, whereas the choice of p guarantees that 0 is an inner point of the triangle U with vertices x_1, x_2, x_3 (see Figure 1). According to (5.1) we have

$$V := \big\{z \in \mathbb{C} : |z-0| \le |z-x_j| \text{ for } j = 1,2,3\big\} \supseteq \mathcal{T}(0).$$

Let ρ be the radius of the smallest ball centered at 0 and containing the whole triangle V. From the definition of $\mathcal{T}(x)$ and $\Delta(\mathcal{T}(x))$ we see that $\Delta(\mathcal{T}(0)) \le 2\rho$.

The vertices of V are the points v_{12}, v_{23}, v_{31} such that

$$|x_i - v_{ij}| = |x_j - v_{ij}| = |0 - v_{ij}|. \tag{5.6}$$

These equations have a unique solution

$$v_{ij} = \mathrm{i}\frac{x_i x_j(\overline{x_i} - \overline{x_j})}{2\Im(x_i\overline{x_j})}, \quad \text{whence} \quad |v_{ij}| = \frac{1}{2}\left|\frac{x_i x_j(x_i - x_j)}{\Im(x_i\overline{x_j})}\right|. \tag{5.7}$$

Then $\rho = \max |v_{ij}|$, thus the estimate (5.5) is valid for $y = 0$ and it remains to show that it is valid for all $y \in \Sigma(\Omega)$. If $y' \in [0, c/2)$ then the three points $y + x_j$ for $j = 1, 2, 3$ are in $\Sigma(\Omega)$. If $y' \in [c/2, c)$ then the three points $y - x_j$ for $j = 1, 2, 3$ are in $\Sigma(\Omega)$. Both of these cases follow from the fact that $x_1', x_2', x_3' \in (0, c/2)$. Therefore either x_1, x_2, x_3 or $-x_1, -x_2, -x_3$ are elements of $\Sigma(\Omega) - y$, which means that the same estimate (5.5) can be used. □

To describe the palette of $\Sigma(\Omega)$, we find all possible L-patches, i.e., the local configurations around the points of $\Sigma(\Omega)$ up to a distance L. More precisely, the *L-patch* of $x \in \Sigma(\Omega)$ is the set

$$\mathcal{P}_L(x) := \big(\Sigma(\Omega) \cap B(x, L)\big) - x. \tag{5.8}$$

Since we consider the window $\Omega = [0, c)$, the L-patch equals

$$\mathcal{P}_L(x) = \{z \in \mathbb{Z}[\gamma] : x' + z' \in [0, c) \text{ and } |z| \leq L\}. \tag{5.9}$$

Lemma 5.2. *Let $x, y \in \Sigma(\Omega)$ with $\Omega = [0, c)$ and L satisfying* (5.4)*. Then the equality of two L-patches $\mathcal{P}_L(x) = \mathcal{P}_L(y)$ implies the equality of the protocells, i.e., $\mathcal{T}(x) - x = \mathcal{T}(y) - y$.*

Proof. Using (5.3) we can write

$$\mathcal{T}(x) = \big\{z \in \mathbb{C} : |z - x| \leq |z - v| \text{ for all } v \in \Sigma(\Omega) \cap B(x, L)\big\}$$

and thus

$$\mathcal{T}(x) - x = \big\{s \in \mathbb{C} : |s| \leq |s - w| \text{ for all } w \in \mathcal{P}_L(x)\big\},$$

which depends only on $\mathcal{P}_L(x)$ and not on x itself. □

Lemma 5.3. *Let $x, y \in \Sigma(\Omega)$ with $\Omega = [0, c)$ and $L > 0$. If $\mathcal{P}_L(x) \neq \mathcal{P}_L(y)$ then there exists ξ from the following finite subset of $[0, c]$:*

$$\Xi := \big\{z' : z \in \mathcal{P}_L(0)\big\} \cup \big\{c - z' : z \in \mathcal{P}_L(0)\big\}, \tag{5.10}$$

such that ξ lies between x' and y', more precisely, $\min\{x', y'\} < \xi \leq \max\{x', y'\}$.

Proof. Without loss of generality, suppose that there exists z such that $z \in \mathcal{P}_L(x)$ and $z \notin \mathcal{P}_L(y)$. According to (5.9) we have $|z| \leq L$, $x' + z' \in [0, c)$, and $y' + z' \notin [0, c)$.

If $x' < y'$ then $x' + z' < c \leq y' + z'$, therefore $0 \leq x' < c - z' \leq y' < c$ and thus x' and y' are separated by $\xi := c - z'$. We have that $c - z' \in (0, c)$, or equivalently $z' \in (0, c)$. As $|z| \leq L$, we conclude that $z \in \mathcal{P}_L(0)$.

If $x' > y'$ then $y' + z' < 0 \leq x' + z'$, therefore $0 \leq y' < -z' \leq x' < c$ and thus x' and y' are separated by $\xi := -z'$. We have that $-z' \in (0, c)$. As $|-z| = |z| \leq L$, we conclude that $-z \in \mathcal{P}_L(0)$. □

The two lemmas enable us to partition the interval Ω into sub-intervals such that the points of $\Sigma(\Omega)$ whose Galois conjugates lie in the same sub-interval have the same protocell, formally:

Corollary 5.4. *Let $\Omega = [0, c)$ be an interval. Then there exists a finite set $\Xi = \{\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = c\}$ such that the mapping*

$$x' \mapsto \mathcal{T}(x) - x$$

is constant on $[\xi_{j-1}, \xi_j) \cap \mathbb{Z}[\gamma']$ for each $j = 1, \ldots, N$.

Proof. Consider L satisfying (5.4) and let Ξ be given by (5.10). Suppose $x, y \in \Sigma(\Omega)$ satisfy $x', y' \in [\xi_{j-1}, \xi_j)$. According to Lemma 5.3 we have $\mathcal{P}_L(x) = \mathcal{P}_L(y)$. Therefore by Lemma 5.2 their protocells are equal. $\square$

Remark 5.5. From the last two statements, we can conclude that $\Sigma([0, c))$ is a *repetitive set with finite local complexity*, i.e., for each $L > 0$, the number of L-patches is finite and each of them appears for infinitely many $x \in \Sigma([0, c))$. Finite local complexity is justified by the finiteness of set Ξ. Repetitiveness is justified by the fact that the each interval $[\xi_{j-1}, \xi_j)$ contains infinitely many points of $\mathbb{Z}[\gamma']$.

Let us mention that while finite local complexity is a property of all cut-and-project sets, repetitiveness depends on the boundary of window Ω. In particular, cut-and-project sets with Ω of the form $[l, r)$ are repetitive, cf. [Moo97].

The corollary is constructive and it allows us to compute all protocells of the Voronoi tessellation of $\Sigma(\Omega)$ for a fixed $\Omega = [0, c)$:

Algorithm 5.6.

- Input: γ satisfying (2.1), $\Omega = [0, c)$, L satisfying (5.4), e.g.given by (5.5).
- Output: The palette of $\Sigma(\Omega)$.

(1) Compute the set $\Xi = \{\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = c\}$ given by (5.10).
(2) For each interval $[\xi_j, \xi_{j+1})$ compute the corresponding L-patch.
(3) Compute the corresponding protocells to each of these patches.
(4) Remove possible duplicates in the list of protocells.

Example 5.7. We illustrate how the algorithm works for $\gamma = \gamma_T$ the complex Tribonacci constant and $c = 2/(1 - \gamma') = \beta^2 + 1$, where we denote as usual $\beta := 1/\gamma'$. In this case, $\Sigma([0, c)) = X^2(\gamma)$ by Theorem 4.1. We have $k = -1$ in Lemma 5.1 and since $\arg \gamma \in (\pi/2, \pi)$, we have $p = 2$. Therefore L is the maximum of the values

$$\frac{1}{|\gamma|}\left|\frac{\gamma(\gamma-1)}{\Im\gamma}\right| \approx 1.877, \quad \frac{1}{|\gamma|}\left|\frac{\gamma^2(\gamma^2-1)}{\Im(\gamma^2)}\right| \approx 1.877, \quad \frac{1}{|\gamma|}\left|\frac{\gamma^2(\gamma-1)}{\Im\gamma}\right| \approx 2.546,$$

i.e., $L = |\gamma(\gamma - 1)|/\Im\gamma$. The set $\{z' : z' \in \mathbb{Z}[\gamma'] \cap [0, c) \text{ and } |z| \leq L\}$ contains 28 points. The set Ξ, given as a union of two 28-element sets in (5.10), has only 33 elements instead of 56 because many elements appear in both of them. This gives

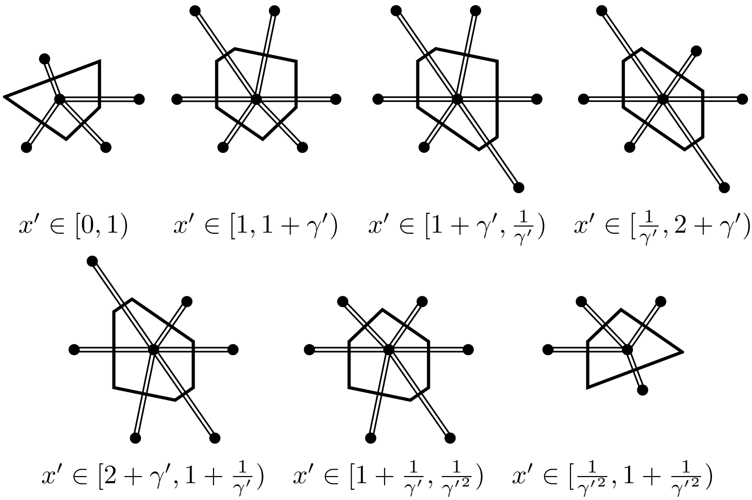


FIGURE 2. Voronoi protocells (the palette) for $X^2(\gamma) = \Sigma(\Omega)$, where $\Omega = [0, \frac{2}{1-\gamma'})$ and $\gamma = \gamma_T$ is the complex Tribonacci constant.

32 cases in steps 2–3 of the algorithm. After we remove the duplicates in the list of the 32 protocells, we end up with the list in Figure 2. The double lines connect the center of the protocell with the centers of the neighboring cells. A part of the Voronoi tessellation of $\Sigma(\Omega)$ is drawn in Figure 3. Note that all computations are performed in the algebraic library of Sage [Sage]. Numbers $a+b\gamma+c\gamma^2 \in \mathbb{Z}[\gamma]$ are stored as triples of integers (a,b,c) and thus results of all arithmetic operations are precise.

Let us determine the parameters $\ell_2(\gamma)$ and $L_2(\gamma)$, with the help of relations (5.2). For each protocell $\mathcal{T}$, the value $\delta(\mathcal{T})$ is by definition the length of the shortest double line in the picture of $\mathcal{T}$. In Figure 4, the 1st protocell is depicted: the neighbors are (counterclockwise) $x_1 = 1$, $x_2 = 2+2\gamma+\gamma^2 = \gamma^{-2}$, $x_3 = 1+\gamma+\gamma^2 = \gamma^{-1}$ and $x_4 = 2+\gamma+\gamma^2 = 1+\gamma^{-1}$. The closest point of these to 0 is $x_2 = \gamma^{-2}$. For the last protocell, the closest point is analogously $-\gamma^2$. Therefore $\delta(\mathcal{T}) = |\gamma^{-2}| = \gamma'$ for the first and the last protocell. For the rest of the protocells, the closest point to 0 is $\pm(1+\gamma+\gamma^2) = \pm\gamma^{-1}$, and therefore $\delta(\mathcal{T}) = |\gamma^{-1}| = \sqrt{\gamma'} = 1/\sqrt{\beta}$. Since $\ell_2(\gamma)$ is the minimum of all $\delta(\mathcal{T})$, we get that

$$\ell_2(\gamma) = \gamma' \approx 0.544.$$

To compute $L_2(\gamma)$, we first determine the value of $\Delta(\mathcal{T})$ for all protocells. By definition, $\Delta(\mathcal{T})$ is twice the maximal distance from 0 to the vertices of $\mathcal{T}$. The vertices of the protocell are points v_{ij} satisfying that $|x_i - v_{ij}| = |x_j - v_{ij}| = |0 - v_{ij}|$, see Figure 4. This is the same condition as (5.6), thus the points v_{ij} are

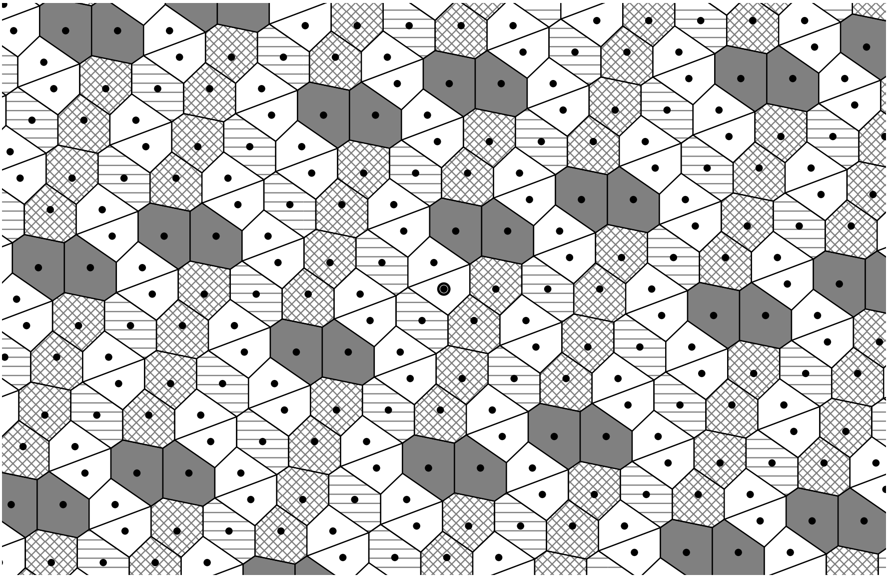


FIGURE 3. Part of the Voronoi tessellation of $X^2(\gamma) = \Sigma(\Omega)$, where $\Omega = [0, \frac{2}{1-\gamma'})$ and $\gamma = \gamma_T$ is the complex Tribonacci constant. The point 0 is highlighted.

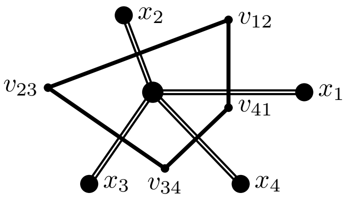


FIGURE 4. One of the protocells of $X^2(\gamma)$.

given by (5.7). Therefore we have

$$|v_{12}| = \frac{1}{2}\Big|\frac{\gamma^{-2}(1-\gamma^{-2})}{\Im(\gamma^{-2})}\Big| \approx 0.692, \quad |v_{23}| = \frac{1}{2}\Big|\frac{\gamma^{-2}(1-\gamma^{-1})}{\Im(\gamma^{-1})}\Big| \approx 0.692,$$

$$|v_{34}| = |v_{41}| = \frac{1}{2}\Big|\frac{\gamma^{-1}(1+\gamma^{-1})}{\Im(\gamma^{-1})}\Big| \approx 0.510.$$

Numerically, it seems that the first two values are equal. To see that this is true, we only have to check that $|1+\gamma^{-1}| = 2|\Re(\gamma^{-1})|$, because $\frac{\Im(z^2)}{\Im z} = 2\Re z$ for any non-real $z \in \mathbb{C}$. Since γ^{-1} and $\overline{\gamma}^{-1}$ are the Galois conjugates of β root of

$Y^3 - Y^2 - Y - 1$, we have $\gamma^{-1}\overline{\gamma}^{-1} = 1/\beta$ and $\gamma^{-1} + \overline{\gamma}^{-1} = 1 - \beta$ by Vieta's formulas. Now we easily verify that the numbers $|1+\gamma^{-1}|^2 = (1+\gamma^{-1})(1+\overline{\gamma}^{-1})$ and $4|\Re(\gamma^{-1})|^2 = (\gamma^{-1}+\overline{\gamma}^{-1})^2$ are equal. We can further simplify

$$|v_{23}|^2 = \frac{1}{4}\frac{\gamma^{-2}\overline{\gamma}^{-2}(1-\gamma^{-1})(1-\overline{\gamma}^{-1})}{\left(\frac{1}{2\mathrm{i}}(\gamma^{-1}-\overline{\gamma}^{-1})\right)^2} = \beta\frac{\beta^2-1}{3\beta^2-1},$$

because we see that the left-hand side is a symmetric rational function in $\gamma^{-1}, \overline{\gamma}^{-1}$, therefore Vieta's formulas can be used to rewrite it in β's.

Whence, for the 1st protocell, the maximal distance is $\Delta(\mathcal{T}) = 2|v_{23}|$. It turns out that this is the value of $\Delta(\mathcal{T})$ for all the protocells of $\Sigma(\Omega)$. Therefore $L_2(\gamma) = \Delta(\mathcal{T}(x))$ for all $x \in X^2(\gamma)$ and the value is

$$L_2(\gamma) = 2\sqrt{\beta\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384.$$

Example 5.8. Let us give one more example. We fix the same $\gamma = \gamma_T$ as before and we take $c = (\gamma')^{-2} = \beta^2$. Then $p = 2$ and $k = 0$ satisfy the hypothesis of Lemma 5.1. Therefore

$$L = \left|\frac{\gamma^2(\gamma-1)}{\Im\gamma}\right| \approx 3.4531$$

satisfies (5.4). In this case, Ξ is of size 40. Figure 5 denotes the result of Algorithm 5.6. We get 7 different protocells. The 4th one has $\delta(\mathcal{T}) = 1$, while all the other ones have $\delta(\mathcal{T}) = \sqrt{\gamma'}$. The value of $\Delta(\mathcal{T})$ is equal to $2\sqrt{\beta\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384$ for all of them.

We can now run Algorithm 5.6 again, using the better upper bound on $\Delta(\mathcal{T})$, namely $L \approx 1.384$. This can save us a lot of steps of the algorithm: The size of Ξ reduces from 40 to 8, so reduces the number of the steps. We will use this improved value of L in Section 6, where we study the sets $\Sigma([0,c))$ for all $c > 0$.

In the two examples, we listed the palettes of $\Sigma([0,c))$ for two different values $c = \beta^2 + 1$ and $c = \beta^2$. Two protocells appears in both lists. The natural question to ask is: For which values of c, a given prototile occurs in the palette of $\Sigma([0,c))$? Using Lemma 5.2, this question can be transformed to an easier one: for which values of c, a specific L-patch occurs in $\Sigma([0,c))$. Since we now treat L-patches for varying c, we denote them $\mathcal{P}_L^c(x)$, and for convenience we denote $\left(\mathcal{P}_L^c(x)\right)' := \{z' : z \in \mathcal{P}_L^c(x)\}$.

Lemma 5.9. *Let $c_0 > 0$ be fixed, $c \in (0, c_0)$ and $L > 0$. Denote $-c_0 =: w_0 < w_1 < \cdots < w_{n-1} < w_n := c_0$ the sequence of numbers such that*

$$W := \{w_1, w_2, \ldots, w_{n-1}\} = \left\{z' \in \mathbb{Z}[\gamma'] : |z| \le L \text{ and } z' \in (-c_0, c_0)\right\}. \tag{5.11}$$

Then

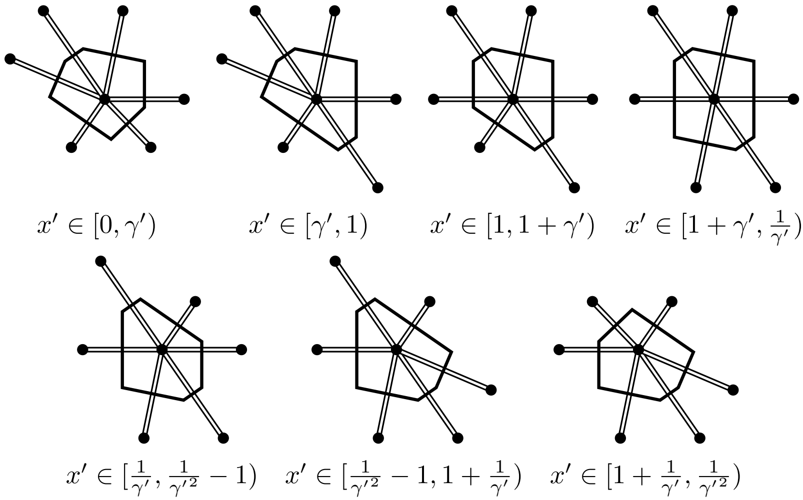


FIGURE 5. Voronoi protocells (the palette) for $\Sigma(\Omega)$, where $\Omega = [0, \frac{1}{\gamma'^2})$ and $\gamma = \gamma_T$ is the complex Tribonacci constant.

(i) *For all* $x \in \Sigma([0,c))$ *we have*

$$\mathcal{P}_L^c(x) \subseteq \{z \in \mathbb{Z}[\gamma] : z' \in W\}.$$

(ii) *For all* $x \in \Sigma([0,c))$ *there exist* $i,k \in \mathbb{N}$, $1 \le i \le k \le n-1$, *such that*

$$\{w_i, w_{i+1}, \dots, w_k\} = \big(\mathcal{P}_L^c(x)\big)'.$$

(iii) *Let* $1 \le i \le k \le n-1$. *Then a finite set* $\{w_i, w_{i+1}, \dots, w_k\}$ *containing* 0 *equals* $(\mathcal{P}_L^c(x))'$ *for some* $x \in \Sigma([0,c))$ *if and only if*

$$w_k - w_i < c < w_{k+1} - w_{i-1}. \tag{5.12}$$

(iv) *For all* $x \in \Sigma([0,c))$ *there exists* $y \in \Sigma([0,c))$ *such that* $\mathcal{P}_L^c(y) = -\mathcal{P}_L^c(x)$.

Proof. (i) As $\Sigma([0,c)) \subseteq \Sigma([0,c_0))$ we have $\mathcal{P}_L^c(x) \subseteq \mathcal{P}_L^{c_0}(x)$ and the statement follows from the relation (5.9).

(ii) Let i and k be the indices for which

$$w_i = \min\big(\mathcal{P}_L^c(x)\big)' \quad \text{and} \quad w_k = \max\big(\mathcal{P}_L^c(x)\big)'.$$

According to the relation (5.9) we get

$$0 \le x' + w_i \quad \text{and} \quad x' + w_k < c. \tag{5.13}$$

Consider w_j for $j \in \mathbb{N}$, $i < j < k$. Then $w_i < w_j < w_k$, whence $0 \le w_j + x' < c$. This implies that w_j belongs to $\big(\mathcal{P}_L^c(x)\big)'$ as well.

(iii) ($\Rightarrow$) Because of (5.13), we have $w_k - w_i < c$. Since w_{i-1} and w_{k+1} do not belong to $\big(\mathcal{P}^c_L(x)\big)'$, we have $x' + w_{i-1} < 0$ and $x' + w_{k+1} \geq c$. Hence $w_{k+1} - w_{i-1} > c$.

(iii) ($\Leftarrow$) Let $w_{i-1}, w_i, w_k, w_{k+1}$ satisfy (5.12). As $\mathbb{Z}[\gamma']$ is dense in $\mathbb{R}$, there exists $u \in (w_{i-1}, w_i)$ such that $u \in \mathbb{Z}[\gamma']$ and $u + c \in (w_k, w_{k+1})$. Put $x' := -u$. Then

$$x' + w_{i-1} < 0 < x' + w_i < x' + w_k < c < x' + w_{k+1}.$$

Since $w_i \leq 0 \leq w_k$, we have that $0 < x' < c$, therefore $x \in \Sigma([0,c))$. We conclude from item (ii) that $\{w_i, w_{i+1}, \ldots, w_k\} = \big(\mathcal{P}^c_L(x)\big)'$.

(iv) Since W is a centrally symmetric set, i.e, $W = -W$, we have that $w_j = w_{n-j}$ for all $0 \leq j \leq n$. Then (5.12) is equivalent to

$$w_{n-i} - w_{n-k} < c < w_{n-i+1} - w_{n-k-1}.$$

According to item (iii), the set $\{w_i, \ldots, w_k\}$ is an L-patch for some $x \in \Sigma([0,c))$ if and only if $\{-w_k, \ldots, -w_i\}$ is an L-patch for some $y \in \Sigma([0,c))$. □

Inequality (5.12) answers our question. To any L-patch, we can assign an open interval such that this patch occurs in $\Sigma([0,c))$ if and only if c lies in this interval. This fact has an important consequence: for any given set of L-patches, the range of c such that these patches are precisely the L-patches of $\Sigma([0,c))$ is an intersection of intervals and complements of intervals. As before, the result on L-patches implies the following result on palettes.

Corollary 5.10. *Let $b_0, c_0 \in \mathbb{R}$ satisfy that $0 < b_0 < c_0$. Denote by $\operatorname{Pal}(\Omega)$ the palette of $\Sigma(\Omega)$, i.e., the set of all protocells of $\Sigma(\Omega)$. Then there exists a finite sequence $b_0 =: \theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N := c_0$ such that the mapping*

$$c \mapsto \operatorname{Pal}\big([0,c)\big)$$

is constant on each of the intervals (θ_{j-1}, θ_j) for $j = 1, \ldots, N$.

Proof. Consider L satisfying (5.4) for $\Sigma = \Sigma([0,b_0))$. For W given by (5.11) find $\theta_1 < \cdots < \theta_{N-1}$ such that

$$\Theta := (W - W) \cap (b_0, c_0) = \{\theta_1, \ldots, \theta_{N-1}\}. \tag{5.14}$$

Let $c, d \in (b_0, c_0)$ and suppose that the palette of $\Sigma([0,c))$ does not coincide with the palette of $\Sigma([0,d))$. Without loss of generality there exists an L-patch of $x \in \Sigma([0,c))$ that is not an L-patch of any $y \in \Sigma([0,d))$. This means that c satisfies inequalities (5.12) for some indices i, k, whereas d does not satisfy them. This fact implies that c and d are separated by a point $w_k - w_i \in W - W$. □

The previous corollary says that there exist only finitely many palettes for $\Sigma([0,c))$ with $c \in [b_0, c_0)$. The following algorithm determines them:

Algorithm 5.11.

- Input: γ satisfying (2.1), $0 < b_0 < c_0$, L satisfying (5.4) for $\Omega = [0, b_0)$, e.g.given by (5.5).
- Output: All possible palettes $\mathrm{Pal}(\Omega)$ of $\Sigma(\Omega)$ for $\Omega = [0, c)$ and $b_0 \le c < c_0$.

(1) Compute the set $\Theta = \{\theta_1 < \cdots < \theta_{N-1}\}$ given by (5.14).
(2) Using Algorithm 5.6, compute the palettes $\mathrm{Pal}(\Omega)$ for all $\Omega = [0, c)$ with $c = b_0, \frac{b_0+\theta_1}{2}, \theta_1, \ldots, \frac{\theta_{N-2}+\theta_{N-1}}{2}, \theta_{N-1}, \frac{\theta_{N-1}+c_0}{2}$.
(3) Remove possible duplicates in the list of palettes.

In Corollary 5.10 and Algorithm 5.11, the assumption $b_0 > 0$ is very important, because there exist infinitely many $c \in (0, c_0)$ with different palettes. However, these palettes cannot differ too much. In fact, the self-similarity property (see Proposition 4.2) guarantees that the palette for the window $[0, \gamma' c)$ differs from the palette for $[0, c)$ only by a scaling factor γ. Therefore the knowledge of the palettes for $c \in [\gamma' c_0, c_0)$ is sufficient for the description of all palettes.

Remark 5.12. As a consequence of item (iv) of Lemma 5.9, the list of L-patches for $\Sigma([0, c))$ is invariant under rotation by 180°. Therefore the palette $\mathrm{Pal}([0, c))$ is invariant as well. Figures 2 and 5 witness this phenomenon.

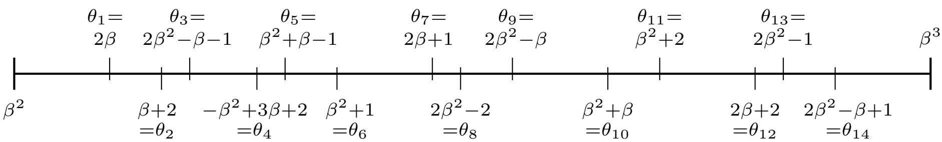
6. Complex Tribonacci number exploited. Proof of Theorem 1.2

In this section, we describe the details of the proposed workflow on an example — the complex Tribonacci base $\gamma = \gamma_T$. We aim at the proof of Theorem 1.2. As usual, $\beta := \gamma\overline{\gamma} = 1/\gamma'$. The theorem will be proved by combining the self-similarity property in Proposition 4.2 and the following result:

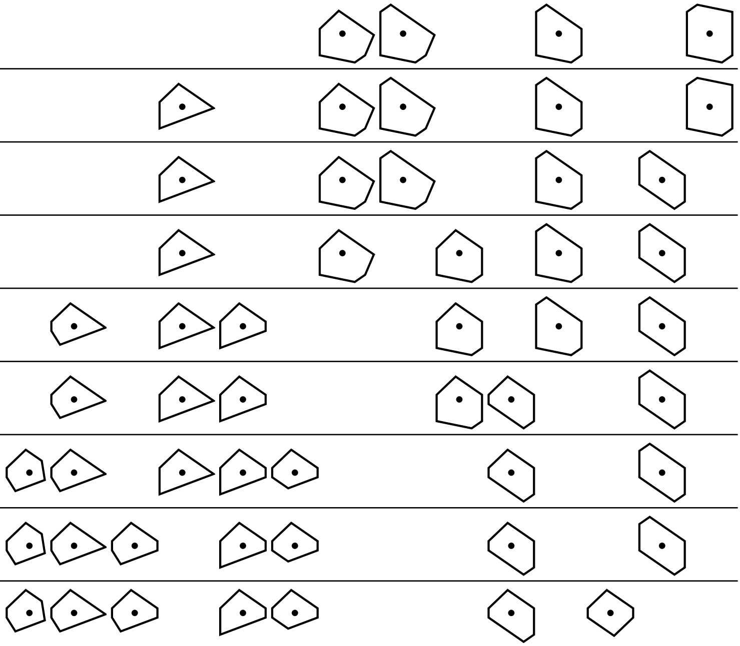
Proposition 6.1. *Let $\Omega = [0, c)$ with $c \in (\beta^2, \beta^3)$, where $\beta := 1/\gamma'$ and γ is the complex Tribonacci constant. Denote $\Sigma := \Sigma(\Omega)$. Then*

$$\min_{x\in\Sigma} \delta\big(\mathcal{T}(x)\big) = 1/\beta \quad \text{and} \quad \max_{x\in\Sigma} \Delta\big(\mathcal{T}(x)\big) = 2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}}. \tag{6.1}$$

Proof. We put $b_0 := \beta^2$ and $c_0 := \beta^3$. In Example 5.8 we have shown that $L = 2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}} \approx 1.384$ satisfies (5.4) for $\Omega = [0, b_0)$. Using this L, we run Algorithm 5.11. The first step of the algorithm computes the set Θ defined by (5.14). This Θ has 14 elements, they are drawn in the following picture:



The number of cases in step 2 of the algorithm is then 30. This means that we have to run Algorithm 5.6 exactly 30 times to obtain all possible palettes. Amongst

Interval for c	The palette of $\Sigma(\Omega)$, where $\Omega = [0, c)$
β^2	
$(\beta^2, 2\beta)$	
$(2\beta, \beta+2)$	
$(\beta+2, \beta^2+1)$	
$(\beta^2+1, 2\beta+1)$	
$(2\beta+1, \beta^2+\beta)$	
$(\beta^2+\beta, \beta^2+2)$	
$(\beta^2+2, 2\beta+2)$	
$(2\beta+2, \beta^3)$	

Tile	$\frac{1}{\gamma}\mathcal{T}_4$	$\mathcal{T}_1$	$\frac{1}{\gamma}\mathcal{T}_5$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\frac{1}{\gamma}\mathcal{T}_8$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$	$\mathcal{T}_8$	$\frac{1}{\gamma}\mathcal{T}_{10}$	$\mathcal{T}_9$	$\mathcal{T}_{10}$
Value of δ	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\beta}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	$\frac{1}{\sqrt{\beta}}$	1
Value of Δ	A	B	A	B	B	A	B	B	B	B	B	A	B	B
Value of Δ^*	1	1	1	1	1	1	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\beta}$	1	$\sqrt{\beta}$	$\sqrt{\beta}$

TABLE 1. The protocells for the complex Tribonacci constant for windows $\Omega = [0, c)$ with $c \in [\beta^2, \beta^3)$. We put $A := 2\sqrt{\frac{\beta^2-1}{3\beta^2-1}}$ and $B := A\sqrt{\beta}$. Each tile in the list appears rotated by 180°as well, we omit these to make the table shorter; see Remark 5.12. For a cut-point θ_i, the palette is the intersection of the palettes for the surrounding intervals, for instance $\mathrm{Pal}([0, \beta^2+1)) = \{\mathcal{T}_2, \mathcal{T}_6, \mathcal{T}_8, \mathcal{T}_9, -\mathcal{T}_8, -\mathcal{T}_6, -\mathcal{T}_2\}$.

the 30 cases mentioned above, there are some duplicates, and we end up with only 16 cases: 8 cases correspond to cut-points $\theta_0, \theta_1, \theta_2, \theta_6, \theta_7, \theta_{10}, \theta_{11}, \theta_{12}$, the other 8 cases correspond to the open intervals between the cut-points. Moreover, we observe that for each cut-point θ_i, the palette $\mathrm{Pal}([0, \theta_i))$ is the intersection of the palettes of the two surrounding intervals. All the palettes for the intervals are depicted in Table 1.

At the bottom of the table, the values of $\delta(\mathcal{T})$ and $\Delta(\mathcal{T})$ are written out for each protocell. It turns out that every row of the table but the special case $c = \beta^2$ has the minimal value of δ equal to $1/\beta \approx 0.5437$ and the maximal value of Δ equal to $2\sqrt{\beta}\sqrt{\frac{\beta^2-1}{3\beta^2-1}} \approx 1.3843$. $\square$

We recall that two of the runs of Algorithm 5.6, for $c = \frac{2}{1-\gamma'} = \beta^2 + 1 \in \Theta$, i.e., for $X^2(\gamma)$, and for $c = \beta^2$ are explained in Examples 5.7 and 5.8 (cf. also Figures 2 and 5). We have drawn a part of the Voronoi tessellation of $X^2(\gamma)$ in Figure 3.

Proof of Theorem 1.2. The theorem is a direct corollary of Proposition 4.2, Theorem 4.1, Proposition 6.1 and of the following two facts:

- It cannot happen that $c = m/(1-\gamma') = (\gamma\overline{\gamma})^k = \beta^k$ for some $m \geq 1$ and $k \in \mathbb{Z}$. For, assume on the contrary that the last equation holds. Then $\beta^k \geq m$ and so $k \geq 1$. Moreover, $k \geq 3$, since γ is cubic, and we have, by Galois isomorphism, that $m\gamma^k = 1-\gamma$. The relation $|m\gamma^k| \geq |\gamma^3| > |1-\gamma|$ yields a contradiction.
- If $\mathcal{T}$ is a Voronoi protocell in $\Sigma(\Omega)$ then $\gamma^k\mathcal{T}$ is a Voronoi protocell in $\gamma^k\Sigma(\Omega) = \Sigma((\gamma')^k\Omega)$ for any $k \in \mathbb{Z}$. For any $m \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ such that $(\gamma')^k\frac{m}{1-\gamma'} \in (\beta^2, \beta^3)$. $\square$

Remark 6.2. Let us point out that for a real base β the characteristic $L_m(\beta)$ given by (1.1) is not influenced by gaps $x_{k+1} - x_k$ occurring only in a bounded piece of the real line. Therefore in general the value $L_m(\gamma)$ as we have defined for the complex number γ is not the precise analogy to $L_m(\beta)$. Nevertheless, if the set $X^m(\gamma)$ is repetitive (i.e., any patch occurs infinitely many times), which is our case, then omitting configurations in a bounded area of the plane plays no role.

7. Delone tessellation — dual to Voronoi tessellation

From Voronoi tessellation we can construct its dual tessellation: Let $\Sigma \subseteq \mathbb{C}$ be a Delone set. Consider a planar graph in $\mathbb{C}$ whose vertices are elements of the set Σ and edges are line segments connecting $x, y \in \Sigma$ where x and y are *neighbors*, i.e., their Voronoi cells $\mathcal{T}(x)$ and $\mathcal{T}(y)$ share a side. This graph divides the complex plane into faces; these faces are called *Delone tiles*. The collection of Delone tiles is the *Delone tessellation* of Σ.

All vertices of a Delone tile lie on a circle; its center is a vertex of the Voronoi tessellation. This is illustrated in Figure 6, which shows a small part of the set $X^2(\gamma)$, where γ is the complex Tribonacci constant; the quadrilateral is inscribed

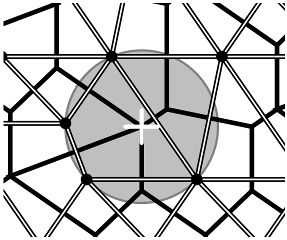


FIGURE 6. Part of Voronoi (in solid lines) and Delone (in double lines) tessellations of $X^2(\gamma)$ for $\gamma = \gamma_T$ the complex Tribonacci constant. The white cross is a vertex of the Voronoi tessellation, and at the same time, it is a center of the gray circle, on which four points of $X^2(\gamma)$ lie.

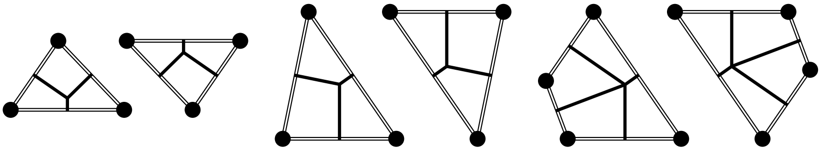


FIGURE 7. Delone tiles of the set $X^2(\gamma)$, where $\gamma = \gamma_T$ is the complex Tribonacci constant.

in the circle. The white cross marks the center of the circle and it is a common vertex of four Voronoi cells.

The minimal distance $\inf_{x\in\Sigma} \delta(\mathcal{T}(x))$ is equal to the shortest edge in the Delone tessellation. On the other hand, the longest edge in the Delone tessellation is (in general) shorter than $\sup_{x\in\Sigma} \Delta(\mathcal{T}(x))$. Therefore, for a point $x \in \Sigma(\Omega)$ we can define

$$\Delta^*\big(\mathcal{T}(x)\big) \coloneqq \max\big\{|x-y| : y \text{ is a neighbor of } x \text{ in } \Sigma\big\}$$

and study its maximum over all points $x \in \Sigma$.

We can apply this to the sets $X^m(\gamma)$. We define

$$L^*_m(\gamma) = L^*_m(\gamma) \coloneqq \sup_{x\in X^m(\gamma)} \Delta^*\big(\mathcal{T}(x)\big)$$

if $X^m(\gamma)$ is Delone, and $L^*_m(\gamma) = +\infty$ otherwise. When $X^m(\gamma)$ is a cut-and-project set, we know that it has a finite local complexity and therefore finitely many different Delone tiles up to translation.

In the case of the complex Tribonacci base, the shapes of all Delone tiles of $X^2(\gamma)$ are depicted in Figure 7. From Table 1 we get the following result:

Theorem 7.1. *With the hypothesis of Theorem 1.2, we have:*

$$L^*_m(\gamma) = |\gamma|^{3-k}.$$

8. Comments and open problems

This paper treated a family of cubic complex Pisot units γ — such ones that the real number $1/\gamma'$ is positive and satisfies Property (F). We used the concept of cut-and-project sets to study the properties of the sets $X^m(\gamma)$. However, there are other cases where it might be possible to use this concept:

(1) We can consider a different perspective of the Tribonacci constant. Let γ be the complex root of $Y^3 + Y^2 + Y - 1$, and put $\beta := 1/\gamma'$. Both γ and $-\gamma$ are complex Pisot units.

It was shown by Vávra [Váv14] that the real Tribonacci constant β has the so-called Property (−F). Shortly speaking, all numbers from $I \cap \mathbb{Z}[-1/\beta] = I \cap \mathbb{Z}[\beta]$, where $I := (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, have a finite expansion of the form $\frac{a_1}{-\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{-\beta^3} + \cdots$ with $a_j \in \{0, 1\}$. From this, we can show that $X^m(-\gamma)$ is a cut-and-project set for arbitrary $m \geq 1$. The idea goes along the lines of the proof of Theorem 4.1.

(2) Consider any real Pisot unit β of degree n. Let $\gamma = \mathrm{i}\sqrt{\beta}$. Then γ is a complex Pisot unit of degree $2n$, its Galois conjugates are $\overline{\gamma}$ and $\pm\mathrm{i}\sqrt{\beta'}$ for β' conjugates of β.

Clearly $X^m(\gamma) = X^m(-\beta) + \mathrm{i}\sqrt{\beta}X^m(-\beta)$. Therefore the Voronoi cells of $X^m(\gamma)$ are rectangles. Values $\ell_m(\gamma)$ and $L_m(\gamma)$ can be easily obtained from the minimal and maximal distances in $X^m(-\beta)$. In the case $n = 2$, relations between $X^m(-\beta)$ and cut-and-project sets in dimensions $d = e = 1$ were established in [MPP14], implying that $X^m(\gamma)$ is related to cut-and-project sets in dimensions $d = e = 2$.

Let us note that Zaïmi [Zaï04] evaluated $\ell_m(\gamma)$ for $\gamma = \mathrm{i}\sqrt{\beta}$, $m = \lfloor \beta^2 \rfloor$ and $\beta > 1$ the root of $Y^2 - aY - a$, $a \in \mathbb{N}$.

(3) In the cubic case, we can weaken the hypothesis of Theorem 4.1. For a fixed m, the Property (F) can be replaced by the assumption that all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have a finite β-representation over the alphabet $\{0, 1, \ldots, m\}$, where we denote $\beta := 1/\gamma' > 1$. Under such an assumption, $X^m(\gamma)$ is a cut-and-project set.

Akiyama, Rao and Steiner [ARS04] described precisely the set of purely periodic expansions of points from $\mathbb{Z}[\beta]$. They have shown that all of them are of the form $.ccc\cdots = .c^\omega$, where $0 \leq c < \lfloor \beta \rfloor$ and $(a + b) \mid c$. Since all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have finite or periodic β-expansions [Sch80] (and the only periods are therefore the ones mentioned above), it is satisfactory to find m_1 such that the number $.(a + b)^\omega$ has a finite representation over the alphabet $\{0, \ldots, m_1\}$. Under this hypothesis, all numbers from $\mathbb{Z}[\beta] \cap [0, 1)$ have a finite representation over the alphabet $\{0, \ldots, m\}$ for all $m \geq m_1 \lfloor \frac{\beta}{a+b} \rfloor$. We were not able to establish the hypothesis in all cases. We list some cases in Table 2.

b	a	m_1	**Representation of** $.(a+b)^\omega$
-2	≥ 3	$2a-2$	$.(a-3)(2a-2)(a-3)(0)(1)$
-3	≥ 7	$3a-6$	$.(a-4)(2a-5)(3a-6)(a-7)(0)(1)$
	$=6$	10	$.(2)(7)(10)(10)(0)(0)(1)$
	$=5$	9	$.(0)(9)(9)(5)(0)(0)(1)$
	$=4$	7	$.(0)(2)(6)(7)(0)^3(1)$
-4	≥ 8	$8a-11$	$.(a-5)(2a-11)(8a-11)(4a-31)(a-8)(0)(1)$
	$=7$	39	$.(0)(16)(39)(27)(0)^3(1)$
	$=6$	47	$.(0)(3)(44)(47)(0)^4(1)$

TABLE 2. List of pairs of a, b such that $X^m(\gamma)$ is a cut-and-project set, where γ is the non-real root of $Y^3 + bY^2 + aY - 1$ and $m \geq m_1 \lfloor \frac{1/\gamma'}{a+b} \rfloor$.

(4) Quartic Pisot units γ with $|\gamma| \in (1,2)$ are treated by Dombek, Masáková and Ziegler in [DMZ13]. The authors study the question of whether every element of the ring $\mathbb{Z}[\gamma]$ of integers of $\mathbb{Q}(\gamma)$ can be written as a sum of distinct units. If the only units on the unit circle are ± 1, then the question can be interpreted as Property (F) over the alphabet $\{-1, 0, 1\}$. Therefore the concept of cut-and-project sets can be applied to these quartic bases and symmetric alphabets as well.

Let us conclude with several open questions:

(A) Is it true that all real cubic Pisot units β with a complex conjugate satisfy the following: There exists $m \in \mathbb{N}$ such that all numbers from $\mathbb{Z}[\beta] \cap [0,1)$ have finite β-representation over the alphabet $\{0, \dots, m\}$?

(B) Which real cubic unit bases $-\beta$, other than minus the Tribonacci constant, satisfy Property ($-$F)? Which $-\beta$ satisfy the statement proposed in Question (A)?

(C) It is well known that, in the real case, $X^m(\beta)$ is a relatively dense set in $\mathbb{R}_+$ if and only if $m > \beta - 1$. Can we state analogous result in the complex case? In particular, is $X^m(\gamma)$ relatively dense set in $\mathbb{C}$ for all $m > |\gamma|^2 - 1$?

Can the complex modification of the Feng's result [Fen13] be proved, namely that $\ell_m(\gamma) = 0$ if and only if $m > |\gamma|^2 - 1$ and γ is not a complex Pisot number?

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BETA-EXPANSIONS OF RATIONAL NUMBERS IN QUADRATIC PISOT BASES

TOMÁŠ HEJDA AND WOLFGANG STEINER

Abstract. We study the purely periodic β-expansions of rational numbers. We give an algorithm for determining the value of the function $\gamma(\beta)$ for quadratic Pisot numbers β. For numbers satisfying $\beta^2 = a\beta + b$ with b dividing a, we show a necessary and sufficient condition for $\gamma(\beta) = 1$, i.e., that all rational numbers $p/q \in [0, 1)$ with $\gcd(q, b) = 1$ have a purely periodic β-expansion.

1. Introduction

Rényi β-expansions [Rén57] provide a very natural generalization of standard positional numeration systems such as the decimal system. Expansions of numbers $x \in [0, 1)$ can be defined in terms of a transformation. Let $\beta > 1$ denote the base. Then the β-transformation is the map

$$T : [0, 1) \to [0, 1),\ x \mapsto \beta x - \lfloor \beta x \rfloor. \tag{1.1}$$

The expansion of x is the infinite string $x_1 x_2 x_3 \cdots$ where $x_j := \lfloor \beta T^{j-1} x \rfloor$. It is a well-known fact that for $\beta \in \mathbb{N}$, the β-expansion of $x \in [0, 1)$ is eventually periodic (i.e., there exists p, n such that $x_{k+p} = x_p$ for all $k \geq n$) if and only if $x \in \mathbb{Q}$. This result was generalized to all Pisot bases by Schmidt [Sch80], who proved that for a Pisot number β the expansion of $x \in [0, 1)$ is eventually periodic if and only if x is an element of the algebraic field $\mathbb{Q}(\beta)$. Moreover, he showed that when β satisfies $\beta^2 = a\beta + 1$, then all $x \in [0, 1) \cap \mathbb{Q}$ have a purely periodic β-expansion.

Akiyama [Aki98] showed that if β is a Pisot unit satisfying a certain finiteness property called (F') then there exists $c > 0$ such that all rational numbers $x \in \mathbb{Q} \cap [0, c)$ have a purely periodic expansion. If β is not a unit, then a rational number $p/q \in [0, 1)$ can have a purely periodic expansion only if q is co-prime to the norm $N(\beta)$. We denote $\mathbb{Z}_b$ the set of rational numbers p/q with $\gcd(q, b) = 1$. Many Pisot non-units satisfy that there exists $c > 0$ such that all $x \in \mathbb{Z}_{N(\beta)} \cap [0, c)$ have purely periodic expansion. This stimulates for the following definition:

Definition 1.1. Let β be a Pisot number, and let $N(\beta)$ denote the norm of β. Then we define $\gamma(\beta) \in [0, 1]$ as the infimum of positive rational numbers $p/q \in \mathbb{Q}$

with $\gcd(q, N(\beta)) = 1$ and with not purely periodic β-expansion:

$$\gamma(\beta) := \inf\big\{\tfrac{p}{q} : p, q > 0,\ \gcd(q, N(\beta)) = 1,\\ \tfrac{p}{q} \text{ does not have a purely periodic } \beta\text{-expansion}\big\}.$$

The question is how to determine the value of $\gamma(\beta)$. As well, knowing when $\gamma(\beta) = 0$ or 1 is of big interest.

The transformation T possesses an ergodic invariant measure. Therefore this transformation on the interval $[0, 1)$ forms a dynamical system. It is easy to observe that the expansion of x is purely periodic if and only if x is a periodic point of T, i.e., there exists $p \geq 1$ such that $T^p x = x$. The natural extension $(\mathcal{X}, \mathcal{T})$ of $([0, 1), T)$ can be defined in an algebraic way, cf. (2.1). Taking this form of the natural extension, several authors contributed to proving the following result: A point $x \in [0, 1)$ has purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and its diagonal embedding lies in the natural extension domain $\mathcal{X}$. The quadratic unit case was solved by Hama and Imahashi [HI97], the confluent unit case by Ito and Sano [IS01, IS02]. Then Ito and Rao [IR05] resolved the unit case completely using an algebraic argument. For non-unit bases β, one has to consider finite (p-adic) places of the field $\mathbb{Q}(\beta)$. This consideration allowed Berthé and Siegel [BS07] to expand the result to all (non-unit) Pisot numbers.

The first values of $\gamma(\beta)$ for two particular non-units were provided by Akiyama, Barat, Berthé and Siegel [ABBS08]. Recently, Minervino and Steiner [MS14] described the boundary of $\mathcal{X}$ for quadratic non-unit Pisot bases. This allowed them to find the value of $\gamma(\beta)$:

Theorem 1.2 ([MS14]). *Let β be the positive root of $\beta^2 = a\beta + b$ for $a \geq b > 0$ two co-prime integers. Then*

$$\gamma(\beta) = \begin{cases} 1 - \frac{(b-1)b\beta}{\beta^2 - b^2} \in (0, 1) & \text{if } a > b(b-1), \\ 0 & \text{otherwise.} \end{cases}$$

2. Preliminaries

2.1. **Combinatorics on words.** We consider both finite and infinite words over a finite alphabet $\mathcal{A}$. The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. An infinite word is *(eventually) periodic* if it is of the form $v(u)^\omega = vuuu\cdots$; $v \in \mathcal{A}^*$ is the pre-period and $u \in \mathcal{A}^* \setminus \mathcal{A}^0$ is the period; if the pre-period is empty, we speak about a *purely periodic word*. The set of all infinite words over $\mathcal{A}$ is denoted $\mathcal{A}^\omega$, and it is equipped with the Cantor topology. A prefix of a (finite or infinite) word w is any finite word v such that w can be written as $w = vu$ for some word u. We denote by $\mathrm{Pref}(\Omega)$ for $\Omega \subseteq \mathcal{A}^\omega$ the set of all finite prefixes of words in Ω.

For a finite word $u = u_0 u_1 \dots u_{k-1}$ and an arbitrary number α we define a natural polynomial representation of the word as

$$P(\alpha, u) \coloneqq \sum_{i=0}^{k-1} u_i \alpha^i.$$

This definition is extended to infinite words by taking a limit if the limit exists.

2.2. **Representation spaces.** We adopt the notation of [MS14], however, we restrict ourselves to β being a quadratic Pisot number. Let $K = \mathbb{Q}(\beta)$. Since β is quadratic, we know that there are exactly two infinite places of K. In one of them, the norm of x is the absolute value $|x|$; in the second one, K', the norm of x is $|x'|$ where $x \to x'$ is the unique non-identical Galois isomorphism of K. Both these places have $\mathbb{R}$ as their completion.

If β is not a unit, then we have to consider finite places of K as well. We put $\mathbb{K}_\mathrm{f} \coloneqq \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$. The convergence in $\mathbb{K}_\mathrm{f}$ can be expressed in terms of β-adic expansions, cf. §2.4. Finally, $\mathbb{K} \coloneqq K \times K' \times \mathbb{K}_\mathrm{f}$ and $\mathbb{K}' \coloneqq K' \times \mathbb{K}_\mathrm{f}$. We define the diagonal embeddings

$$\delta : \mathbb{Q}(\beta) \to \mathbb{K},\ x \mapsto (x, x', x_\mathrm{f}) \quad \text{and} \quad \delta' : \mathbb{Q}(\beta) \to \mathbb{K}',\ x \mapsto (x', x_\mathrm{f}),$$

where x_f is the vector of the embeddings of x into the spaces $K_\mathfrak{p}$. As well, we define the projections $\pi_1 : \mathbb{K} \to K$, $\pi_2 : \mathbb{K} \to K'$ and $\pi_3 : \mathbb{K} \to \mathbb{K}_\mathrm{f}$. We put $P'(u) = P(\beta', u)$ and $P_\mathrm{f}(u) = P(\beta, u)_\mathrm{f}$ for every word u.

We fix some more notation. For rational integers $a, b \in \mathbb{Z}$, $(a, b) \neq (0, 0)$ we denote by $a \perp b$ the fact that a and b are co-prime, i.e., that $\gcd(a, b) = 1$. Moreover, for $b \neq 0$ we put $(b)^\perp \coloneqq \{a \in \mathbb{Z} : a \perp b\}$ (the set of integers co-prime to b), and $\mathbb{Z}_b \coloneqq \{p/q : p \in \mathbb{Z}, q \in (b)^\perp\}$ (the set of rational numbers with denominator co-prime to b).

2.3. **Beta-tiles.** For $x \in [0, 1)$, we define the β-tile of x as the Hausdorff limit

$$\mathcal{Q}(x) \coloneqq \lim_{i \to \infty} \delta'\big(x - \beta^k T^{-k}(x)\big) \subseteq \mathbb{K}'.$$

Note that the standard definition of a β-tile for $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$ is $\mathcal{R}(x) \coloneqq \delta'(x) - \mathcal{Q}(x)$. We now describe the natural extension $(\mathcal{X}, \mathcal{T})$ of the dynamical system $([0, 1), T)$ as a subset of the representation space $\mathbb{K}$. For quadratic Pisot β, root of $\beta^2 = a\beta + b$ with $a \geq b \geq 1$, it comprises of two suspensions of β-tiles:

$$\mathcal{X} \coloneqq \big([0, \beta - a) \times \mathcal{Q}(0)\big) \cup \big([\beta - a, 1) \times \mathcal{Q}(\beta - a)\big),$$

$$\mathcal{T} : \mathcal{X} \to \mathcal{X},\ \boldsymbol{z} \mapsto \beta \boldsymbol{z} - \delta\big(\lfloor \beta \pi_1(\boldsymbol{z}) \rfloor\big). \tag{2.1}$$

It is remarkable that the natural extension given by this formula is not a closed set, for with the given definition, the following important result holds:

Theorem 2.1 ([HI97, IR05, BS07])**.** *For a Pisot number β, we have that x has a purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and $\delta(x) \in \mathcal{X}$.*

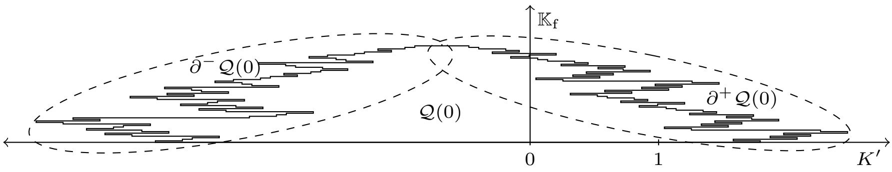

FIGURE 1. The two boundaries of the tile $\mathcal{Q}(0)$ for $\beta = 1 + \sqrt{3}$.

2.4. **Hensel expansions of quadratic numbers.** Throughout the rest of the paper, we will fix arbitrary quadratic Pisot number β, root of $\beta^2 = a\beta + b$ with $a \geq b \geq 1$.

The map P_f is a homeomorphism (a bijection that is continuous both ways) from $\mathcal{A}^\omega$ to $\overline{\mathbb{Z}_b[\beta]_\mathrm{f}}$, where the alphabet is $\mathcal{A} := \{0, 1, \ldots, |N(\beta)| - 1\}$. Its inverse is the *Hensel expansion map* $\boldsymbol{h} : \overline{\mathbb{Z}_b[\beta]_\mathrm{f}} \to \mathcal{A}^\omega$, whose fundamental property is that for $x \in \mathbb{Z}_b[\beta]$, the Hensel expansion $\boldsymbol{h}(x) = x_0 x_1 x_2 \cdots$ satisfies that

$$x - \sum_{i=0}^{n} x_i \beta^i \in \beta^n \mathbb{Z}_b[\beta] \quad \text{for all} \quad n \geq 0. \tag{2.2}$$

3. BETA-TILES AND THE VALUE $\gamma(\beta)$

In §9.3 of the article [MS14], the boundary of β-tiles $\mathcal{Q}(0)$ and $\mathcal{Q}(\beta - a)$ is described. The tiles have two boundaries: $\partial^+ \mathcal{Q}(x)$ on the right and $\partial^- \mathcal{Q}(x)$ on the left (see Figure 1). We have

$$\begin{aligned} \partial^+ \mathcal{Q}(0) = \partial^+ \mathcal{Q}(\beta - a) &= \big\{ \big(1 + P'(\boldsymbol{u}), 1_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega \big\}, \\ \partial^- \mathcal{Q}(0) &= \big\{ \big(\beta' - a + P'(\boldsymbol{u}), \beta_\mathrm{f} - a_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega \big\}, \\ \partial^- \mathcal{Q}(\beta - a) &= \big\{ \big(\beta' - a + 1 + P'(\boldsymbol{u}), \beta_\mathrm{f} - a_\mathrm{f} + 1_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega \big\} \end{aligned} \tag{3.1}$$

(all these sets lie in $\mathbb{K}'$). We can express the value of $\gamma(\beta)$ easily in terms of the boundaries:

Theorem 3.1. *Let β be a quadratic Pisot number. Denote $Y' := K' \times (\mathbb{Z})_\mathrm{f} \subseteq \mathbb{K}'$ and put*

$$\bar{\gamma} := \inf \pi_2\big(\partial^+ \mathcal{Q}(0) \cap Y'\big). \tag{3.2}$$

If $\sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) > 0$*, then* $\gamma(\beta) = 0$.
Otherwise if $\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) > \beta - a$*, then* $\gamma(\beta) = \min\{\beta - a, \max\{\bar{\gamma}, 0\}\}$.
Otherwise $\gamma(\beta) = \max\{\bar{\gamma}, 0\}$.

The decision tree of the theorem is summarized in Table 1

Proof. By Definition 1.1 and Theorem 2.1 we have that

$$\gamma(\beta) = \inf\big\{x \in \mathbb{Z}_b : x \geq 0, \, \delta(x) \notin \mathcal{X}\big\}.$$

	$A \le 0$		$A > 0$	
	$B \le \beta - a$	$B > \beta - a$	$B \le \beta - a$	$B > \beta - a$
$\bar\gamma < 0$	0	0	0	0
$0 \le \bar\gamma < \beta - a$	$\bar\gamma$	$\bar\gamma$	0	0
$\beta - a \le \bar\gamma \le 1$	$\bar\gamma$	$\beta - a$	0	0

TABLE 1. To Theorem 3.1, the dependance of $\gamma(\beta)$ on $A \coloneqq \sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big)$ and $B \coloneqq \sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big)$.

The set $\{\delta(x) : x \in \mathbb{Z}_b\}$ is dense in Y' by [ABBS08, Lemma 4.7].

We see that the set $\{x \in \mathbb{Z}_b : \delta(x) \notin \mathcal{X}\}$ is dense in $(\bar\gamma, +\infty)$, therefore if $\bar\gamma \le 0$, we have that $\gamma(\beta) = 0$, and $\gamma(\beta) \le \bar\gamma$ otherwise.

If $A \coloneqq \sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) > 0$, then the points $\{x \in \mathbb{Z}_b : \delta(x) \notin \mathcal{X}\}$ are dense in the interval $[0, A)$, therefore $\gamma(\beta) = 0$.

Consider now that $A < 0$. If $\bar\gamma < \beta - a$ then $\{\delta(x) : x \in \mathbb{Z}_b \cap (0, \bar\gamma)\} \subseteq \mathcal{X}$ therefore (since we know that $\gamma(\beta) \le \bar\gamma$) $\gamma(\beta) = \bar\gamma$.

If $\bar\gamma > \beta - a$, certainly $\gamma(\beta) \ge \beta - a$. If $B \coloneqq \sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) > \beta - a$, then the points $\{x \in \mathbb{Z}_b : \delta(x) \notin \mathcal{X}\}$ are dense in the interval $(\beta - a, B)$, therefore $\gamma(\beta) = \beta - a$. If $B < 0$, we get that $\{\delta(x) : x \in \mathbb{Z}_b \cap (\beta - a, \bar\gamma)\} \subseteq \mathcal{X}$, therefore $\gamma(\beta) = \bar\beta$. □

Remark 3.2. We can change $\mathbb{Z}$ in the statement of the theorem to $\mathbb{Z}_b$ or to $\mathbb{Z}_b \cap [c, d]$ for arbitrary $c < d$ since we have that $\overline{(\mathbb{Z})_{\mathrm{f}}} = \overline{(\mathbb{Z}_b)_{\mathrm{f}}} = \overline{(\mathbb{Z}_b \cap [c, d])_{\mathrm{f}}}$.

Proof. We have that $\overline{(\mathbb{Z}_b)_{\mathrm{f}}} = \overline{(\mathbb{Z}_b \cap [c, d])_{\mathrm{f}}}$ by [ABBS08, Lemma 4.7]. Clearly $\mathbb{Z} \subseteq \mathbb{Z}_b$. We will prove that $\mathbb{Z}_b \subseteq \overline{(\mathbb{Z})_{\mathrm{f}}}$. Let $x/q \in \mathbb{Z}_b$ be given by $x \in \mathbb{Z}$ and $q \in (b)^\perp$. For each $n \in \mathbb{N}$, the multiple inverse $q^{-1} \bmod b^n \in \mathbb{Z}$ exists. Then

$$\Big|\frac{x}{q} - (q^{-1} \bmod b^n)x\Big|_{\mathfrak{p}} = \frac{1}{|q|_{\mathfrak{p}}}\big|x - (qq^{-1} \bmod b^n)x\big|_{\mathfrak{p}} \le 1|x|_{\mathfrak{p}}|b^n|_{\mathfrak{p}} \le |b|_{\mathfrak{p}}^n \to 0$$

for all $\mathfrak{p} \mid (\beta)$, therefore $(q^{-1}x \bmod b^n)_{\mathrm{f}} \to (x/q)_{\mathrm{f}}$. □

For many cases, we obtain the following direct formula:

Corollary 3.3. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ for $a \ge b \ge 2$. Suppose $a > \frac{1+\sqrt5}{2}b$ or $a = b$ or $a \perp b$. Then*

$$\gamma(\beta) = \max\big\{0, \inf \pi_2\big(\partial^+ \mathcal{Q}(x) \cap Y'\big)\big\}. \tag{3.3}$$

Proof. Case $a > \frac{1+\sqrt5}{2}b$. According to (3.5) for $n = 0$ we have that

$$\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) \le \beta' - a + 1 + \frac{b - 1}{1 - (\beta')^2}.$$

We will show that the right-hand side of this equation is less than $\beta - a$. First, we derive, using $(\beta')^2 = a\beta' + b$, $\beta = a - \beta'$ and $1 - (\beta')^2 > 0$, that it is equivalent to

$$-a - ab - \beta'(a^2 + a + 2b - 2) < 0. \tag{3.4}$$

We know that $\beta < a+1$, therefore $-\beta' = \frac{b}{\beta} > \frac{b}{a+1}$, $\beta = a - \beta' > \frac{a(a+1)+b}{a+1}$ and $-\beta' = \frac{b}{\beta} < \frac{(a+1)b}{a^2+a+b}$. As well, $a^2 + a + 2b - 2 > 0$, therefore we estimate

$$-a - ab - \beta'(a^2 + a + 2b - 2) < \frac{-ab^2((\frac{a}{b})^2 - \frac{a}{b} - 1) - b^2((\frac{a}{b})^2 + 2\frac{a}{b} - 2) - 2b}{a^2 + a + b}.$$

When $\frac{a}{b} > \frac{1+\sqrt{5}}{2}$, all three terms in the numerator are negative. This justifies that

$$\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) \le \beta' - a + 1 + \frac{b-1}{1-(\beta')^2} < \beta - a.$$

Because $\beta - a < 1$, it follows easily that

$$\sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) \le \beta' - a + \frac{b-1}{1-(\beta')^2} < 0.$$

Theorem 3.1 then implies the relation (3.3).

Case $a = b$. Take $a = b > 3$. Then $b = (\beta')^2 + (b-1)(\beta')^3 + (2b+1)(\beta')^4$, therefore $\boldsymbol{h}(b) \in 001(b{-}1)\mathcal{A}^\omega$. We know that there exist letters $a_0, a_1, \dots \in \mathcal{A}$ such that $001(b{-}1)0a_00a_1\dots0a_n \in \mathcal{L}_0$ for all n. Therefore

$$\begin{aligned}\bar{\gamma} &\le 1 + P'(001(b{-}1)0a_00a_1\dots0a_n) + (\beta')^{2n}\tfrac{b-1}{1-(\beta')^2}\\ &\le 1 + P'(001(b{-}1)) + (\beta')^{2n}\tfrac{b-1}{1-(\beta')^2} \xrightarrow{n\to\infty} 1 + P'(001(b{-}1))\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1 + (\beta')^2 + (b-1)(\beta')^3.\end{aligned}$$

For $a = b > 4$, we use the estimate $-\beta' \in (\frac{b}{b+1}, 1)$ to obtain that the right-hand side is $< 2 - \frac{b^3(b-1)}{(b+1)^3}$. This means that $\bar{\gamma} < 0$, therefore $\gamma(\beta) = 0$. For $a = b = 4$, we verify that the right-hand side is negative, namely ≈ -0.0193.

When $a = b = 3$, we verify that $\boldsymbol{h}(21) \in 001200020201\mathcal{A}^\omega$ and using similar arguments as above we obtain

$$\bar{\gamma} \le 1 + P'(001200020201) \approx -0.0726 < 0,$$

therefore $\gamma(\beta) = 0$.

When $a = b = 2$, we can follow the lines of the proof of the case $a > \frac{1+\sqrt{5}}{2}b$, because we observe that (3.4) is satisfied, namely the left-hand side is ≈ -0.1436.

Case $a \perp b$. This case is proved in [MS14, §9]. □

For the boundary, we observe the following, which follows from the fact that the boundary is continuous as a function from $\overline{\mathbb{Z}[\beta]_{\mathrm{f}}} \to K'$:

Lemma 3.4. *For every $n \in \mathbb{N}$, we have that each of the boundaries $\partial^{\pm}(x)$ for $x \in \{0, \beta - a\}$ is contained in a union of rectangles,*

$$\partial^+ \mathcal{Q}(x) \subset \bigcup_{w \in \mathcal{A}^n} \big(1 + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(1_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big),$$

$$\partial^- \mathcal{Q}(0) \subset \bigcup_{w \in \mathcal{A}^n} \big(\beta' - a + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(\beta_{\mathrm{f}} - a_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big), \tag{3.5}$$

$$\partial^- \mathcal{Q}(\beta - a) \subset \bigcup_{w \in \mathcal{A}^n} \big(\beta' - a + 1 + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(\beta_{\mathrm{f}} - a_{\mathrm{f}} + 1_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big).$$

Proof. Follows directly from (3.1). For each $n \in \mathbb{N}$ and each $\boldsymbol{u} \in \mathcal{A}^{\omega}$ we have that

$$P'\big(w(0(b{-}1))^{\omega}\big) \le P'(\boldsymbol{u}) \le P'\big(w((b{-}1)0)^{\omega}\big) \quad \text{for } n \text{ even},$$
$$P'\big(w((b{-}1)0)^{\omega}\big) \le P'(\boldsymbol{u}) \le P'\big(w(0(b{-}1))^{\omega}\big) \quad \text{for } n \text{ odd},$$

where w is a prefix of $\boldsymbol{u}$ of length n, therefore $P'(\boldsymbol{u})$ belongs to the interval with endpoints

$$P'\big(w(0(b{-}1))^{\omega}\big) = (\beta')^{n+1}\tfrac{b-1}{1+(\beta')^2} \quad \text{and} \quad P'\big(w((b{-}1)0)^{\omega}\big) = (\beta')^{n}\tfrac{b-1}{1+(\beta')^2}. \qquad \square$$

Proposition 3.5. *Let $\mathcal{L}_y$ for $y \in \mathbb{Z}[\beta]$ be the language of prefixes of Hensel expansions of numbers from the set $\mathbb{Z} - y$, i.e., $\mathcal{L}_y := \mathrm{Pref}\{\boldsymbol{h}(k - y) : k \in \mathbb{Z}\}$. Then for each $n \in \mathbb{N}$ and $x \in \{0, \beta - a\}$ we can estimate*

$$\inf \pi_2\big(\partial^+ \mathcal{Q}(x) \cap Y'\big) \in 1 + \min\big\{P'(w) : w \in \mathcal{L}_0 \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1],$$

$$\sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) \in \beta' - a + \max\big\{P'(w) : w \in \mathcal{L}_{\beta} \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1], \tag{3.6}$$

$$\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) \in \beta' - a + 1 + \max\big\{P'(w) : w \in \mathcal{L}_{\beta} \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1].$$

Proof. First, we observe that since $\overline{(\mathbb{Z})_{\mathrm{f}}} = \overline{(\mathbb{Z}_b)_{\mathrm{f}}}$, we have that $\mathcal{L}_y = \mathrm{Pref}\{\boldsymbol{h}(z - y) : z \in \mathbb{Z}_b\}$. Let $z \in \overline{(\mathbb{Z})_{\mathrm{f}}}$ be the β-adic height at which the infimum is attained, i.e., such that $\big(\inf \pi_2(\partial^+ \mathcal{Q}(x) \cap Y'), z\big) \in \partial^+ \mathcal{Q}(x)$. $\square$

The right-hand sides of (3.6) are itervals whose lengths shrink exponentially as $n \to \infty$. The only remaining step is to construct the languages $\mathcal{L}_x \cap \mathcal{A}^n$, which is solved by the following statement:

Proposition 3.6. *Let $x, z \in \mathbb{Z}[\beta]$ satisfy that $x - z \in b^n\mathbb{Z}$ for some $n \in \mathbb{N}$. Then the Hensel expansions $\boldsymbol{h}(x)$ and $\boldsymbol{h}(z)$ have a common prefix of the length at least n.*

Therefore all elements of $\mathcal{L}_y$ of the length n are precisely

$$\mathcal{L}_y \cap \mathcal{A}^n = \mathrm{Pref}\big\{\boldsymbol{h}(k - y) : k \in \{0, 1, \dots, b^n - 1\}\big\} \cap \mathcal{A}^n, \tag{3.7}$$

$a/b=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b=1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1
4	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
5	0	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1
6	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
7	0	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1
8	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1
9	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1
10	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1
11	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1
12	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1

TABLE 2. The values of $\gamma(\beta)$ for the case when b divides a. The star '$\star$' means that the value is strictly between 0 and 1.

whence

$$\#(\mathcal{L}_y \cap \mathcal{A}^n) \leq b^n. \tag{3.8}$$

Proof. Since $b = \beta^2 - a\beta \in \beta\mathbb{Z}[\beta]$, we have that $x - z \in \beta^n\mathbb{Z}[\beta]$. Let $\boldsymbol{h}(x) = x_0x_1\cdots x_{n-1}\cdots$. Then $x - \sum_{j=0}^{k-1} x_j\beta^j \in \beta^n\mathbb{Z}[\beta]$ and therefore $z - \sum_{j=0}^{k-1} x_j\beta^j \in \beta^n\mathbb{Z}[\beta]$, which means that $x_0\cdots x_{n-1}$ is a prefix of $\boldsymbol{h}(z)$. □

4. THE CASE b DIVIDES a

In the particular case when b divides a, the structure of $\mathcal{L}_y$ is even simpler, namely we have that $\#(\mathcal{L}_y \cap \mathcal{A}^n) = b^{\lceil n/2\rceil}$, therefore $\#(\mathcal{L}_y \cap \mathcal{A}^{2n}) = \#(\mathcal{L}_y \cap \mathcal{A}^{2n-1})$. This is given by the fact that in this case, $b^k\mathbb{Z}[\beta] = \beta^{2k}\mathbb{Z}[\beta]$. The result for this case can be stated as follows:

Theorem 4.1. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a \geq b \geq 2$ and $\frac{a}{b} \in \mathbb{Z}$.*

We have that $\gamma(\beta) = 1$ if and only if $a \geq b^2$ or $(a, b) \in \{(6, 24), (6, 30)\}$.

If $a = b \geq 3$ then $\gamma(\beta) = 0$.

If $b \leq a \leq b(b-1)$ then $\gamma(\beta)$ can be computed with arbitrary precision.

The two cases $\beta^2 = 24\beta + 6$ and $\beta^2 = 30\beta + 6$ are very exceptional. It is given by the fact that for them, we have that $b - (a/b)$ divides b, which is an important ingredient in their strangeness. Table 2 shows whether $\gamma(\beta)$ is 0, 1 or strictly in between, for $b \leq 12$ and $a/b \leq 15$. The first non-trivial values are listed in Table 3.

Proposition 4.2. *Let β be the dominant root of $\beta^2 = cb\beta + b$ with $b \geq 2$ and $c \geq 1$. Let $x, z \in \mathbb{Z}[\beta]$ satisfy that $x - z \in b^n\mathbb{Z}$ for some $n \in \mathbb{N}$. Then the Hensel expansions $\boldsymbol{h}(x)$ and $\boldsymbol{h}(z)$ have a common prefix of the length at least $2n$.*

a	b	$\gamma(\beta)$
2	2	$0.914803044196\cdots$
6	3	$0.992963560101\cdots$
8	4	$0.933542944675\cdots$
12	4	$0.999897789000\cdots$
10	5	$0.834150794175\cdots$
15	5	$0.995306723671\cdots$
20	5	$0.999999907110\cdots$

a	b	$\gamma(\beta)$
12	6	$0.736114178272\cdots$
18	6	$0.993897266395\cdots$
14	7	$0.584906533458\cdots$
21	7	$0.944526094618\cdots$
28	7	$0.997984788082\cdots$
35	7	$0.999986041767\cdots$
42	7	$0.99999999999971\cdots$

TABLE 3. Numerical values of $\gamma(\beta)$, where $\beta^2 = a\beta + b$, that correspond to the first couple '$\star$' in Table 2.

Therefore all elements of $\mathcal{L}_y$ of the length $2n$ are precisely

$$\mathcal{L}_y \cap \mathcal{A}^{2n} = \mathrm{Pref}\big\{\boldsymbol{h}(k-y) : k \in \{0,1,\dots,b^n-1\}\big\} \cap \mathcal{A}^{2n} \tag{4.1}$$

and

$$\#(\mathcal{L}_y \cap \mathcal{A}^{2n}) = \#(\mathcal{L}_y \cap \mathcal{A}^{2n-1}) = b^n. \tag{4.2}$$

Proof. We have that $\beta^2 = b(c\beta+1) \in b\mathbb{Z}[\beta]$ and $b = \beta^2 - c(1+cb)\beta^3 + c\beta^4 \in \beta^2 + \beta^3\mathbb{Z}[\beta] \subseteq \beta^2\mathbb{Z}[\beta]$, whence $\beta^2\mathbb{Z}[\beta] = b\mathbb{Z}[\beta]$ and also $\beta^{2n}\mathbb{Z}[\beta] = b^n\mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$. By the same argument as in the proof of Proposition 3.6 we obtain (4.1), therefore $\#(\mathcal{L}_y \cap \mathcal{A}^{2n}) \le b^n$.

We will show that the equality is true by showing that $\#(\mathcal{L}_y \cap \mathcal{A}^{2n+2}) \ge b\#(\mathcal{L}_y \cap \mathcal{A}^{2n+2})$.

Let $\boldsymbol{h}(x) = x_0x_1\dots x_{2n}x_{2n+1}x_{2n+2}\dots$ and $\boldsymbol{h}(z) = x_0x_1\dots x_{2n}z_{2n+1}z_{2n+2}\dots$ be the Hensel expansions of x and z, respectively. Put

$$\tilde{x} = \frac{1}{\beta^{2n}}\Big(x - \sum_{j=0}^{2n-1} x_j\beta^j\Big) \quad\text{and}\quad \tilde{z} = \frac{1}{\beta^{2n}}\Big(z - \sum_{j=0}^{2n-1} x_j\beta^j\Big).$$

Let $m = (z-x)/b^n$. Then $\tilde{z} - \tilde{x} = \frac{mb^n}{\beta^{2n}} \in m + \beta\mathbb{Z}[\beta]$. Since x_{2n} and z_{2n} are the first digits of expansion of $\tilde{x}$ and $\tilde{z}$, repsectively, we obtain that $z_{2n} - x_{2n} \equiv m \pmod b$. Since m can take any integer value, all the strings $x_0x_1\dots x_{2n-1}a$ for $a \in \mathcal{A}$ are in $\mathcal{L}_y$. Therefore each element of $\mathcal{L}_y \cap \mathcal{A}^{2n}$ has exactly b prolongations and $\#(\mathcal{L}_y \cap \mathcal{A}^{2n+1}) = b\#(\mathcal{L}_y \cap \mathcal{A}^{2n})$. Whence and $\#(\mathcal{L}_y \cap \mathcal{A}^{2n+2}) \ge \#(\mathcal{L}_y \cap \mathcal{A}^{2n+1}) = b\#(\mathcal{L}_y \cap \mathcal{A}^{2n})$. ☐

Lemma 4.3. *Let $\beta^2 = a\beta + b$ with $a \ge b \ge 1$. Then*

$$(b-1) + (a-b+1)\beta' > 0 \quad \textit{if and only if} \quad a < b^2.$$

Proof. If $a \ge b^2$ then $\frac{1}{\beta} < \frac{1}{a} \le \frac{1}{b^2}$. Together with the fact that $\beta' = \frac{b}{\beta}$ we get that

$$(b-1)-(a-b+1)\beta' = (b-1)-(a-b+1)\frac{b}{\beta} < (b-1)-(b^2-b+1)\frac{b}{b^2} = -\frac{1}{b} < 0.$$

If $a \le b^2-1$ then $\frac{1}{\beta} > \frac{1}{a+1} \ge \frac{1}{b^2}$. Whence

$$(b-1)-(a-b+1)\beta' = (b-1)-(a-b+1)\frac{b}{\beta} > (b-1)-(b^2-1-b+1)\frac{b}{b^2} = 0.$$

This completes the proof. □

Proposition 4.4. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $b \ge 2$, $\frac{a}{b} \in \mathbb{Z}$ and $a \ge b^2$. Then $\gamma(\beta) = 1$.*

Proof. Since $\boldsymbol{h}(0) = 0^\omega$, we have that $\boldsymbol{h}(\mathbb{Z}) \subseteq 0^\omega \cup \bigcup_{n\ge0}(00)^n\{1,\dots,b-1\}\mathcal{A}^\omega$ Using Corollary 3.3 and Proposition 3.5 we obtain that

$$\begin{aligned}\gamma(\beta) \ge \inf \pi_2\big(\partial^+\mathcal{Q}(0)\cap Y'\big) &= 1+\inf\big\{P'(w) : w \in \boldsymbol{h}(\mathbb{Z})\big\}\\ &\ge 1+\inf_{n\ge0}\inf\big\{P'(w) : w \in (00)^n\{1,\dots,b-1\}\mathcal{A}^\omega\big\}\\ &= 1+\inf_{n\ge0}(\beta')^{2n}\inf\big\{P'(w) : w \in \{1,\dots,b-1\}\mathcal{A}^\omega\big\}.\end{aligned}$$

With the help of Lemma 4.3 and since $\beta' < 0$, we compute

$$\begin{aligned}\inf\big\{P'(w) : w \in \{1,\dots,b-1\}\mathcal{A}^\omega\big\} &= P'\big(1(b{-}1)(0(b-1))^\omega\big)\\ &= 1+\frac{(b-1)\beta'}{1-(\beta')^2} = 1+\frac{(b-1)\beta'}{-a\beta'+(b-1)} \ge 1+\frac{(b-1)\beta'}{(1-b)\beta'} = 0.\end{aligned}$$

Altogether, $\gamma(\beta) \ge 1+\inf_{n\ge0} 0 = 1$. From the definition we have that $\gamma(\beta) \le 1$, therefore $\gamma(\beta) = 1$ as desired. □

Proposition 4.5. *Let β_1 and β_2 be the Pisot roots of $\beta_1^2 = 24\beta_1 + 6$ and $\beta_2^2 = 30\beta_2 + 6$, respectively. Then $\gamma(\beta_1) = \gamma(\beta_2) = 1$.*

Proposition 4.6. *Let β be a quadratic Pisot number, root of $\beta^2 = cb\beta + b$ with $b \ge 2$ and $1 \le c < b$. Suppose $(b,c) \ne (6,4)$ and $(b,c) \ne (6,5)$. Then $\gamma(\beta) < 1$.*

Proof. Assume that $1 \le c < b$, with $c \notin \{4,5\}$ if $b = 6$, and let $k = \lceil c/(b-c)\rceil$. The Hensel expansion $\boldsymbol{h}(b^k)$ starts with $0^{2k}1(kb{-}kc)$. If $c/(b-c) \notin \mathbb{Z}$, then we have $k(b-c) > c$ and thus $1+(kb-kc)\beta' \le 1+(c+1)\beta' < 0$, using that $\beta' = -b/\beta < -b/(cb+1) \le -1/(c+1)$. This proves that $\gamma(\beta) < 1$ if c is not a multiple of $b-c$.

Assume now that $c/(b-c) \in \mathbb{Z}$, i.e., $k = c/(b-c)$. Then we have

$$\begin{aligned}b^k &\in \beta^{2k}\Big(1-kc\beta+\tbinom{k+1}{2}c^2\beta^2-\tbinom{k+2}{3}c^3\beta^3+\beta^4\mathbb{Z}[\beta]\Big)\\ &= \beta^{2k}\Big(1+c\beta+\tbinom{k+1}{2}c^2\beta^2-\tbinom{k+2}{3}c^3\beta^3-k\beta^3+\beta^4\mathbb{Z}[\beta]\Big),\end{aligned}$$

where we have used that $kb\beta - k\beta^3 \in \beta^4\mathbb{Z}[\beta]$, thus $b^{k+1} \in \beta^{2k+1}(1+\beta^2\mathbb{Z}[\beta])$, and

$$b^k - \binom{k+1}{2}c^2 b^{k+1} \in \beta^{2k}\Big(1 + c\beta - \binom{k+2}{3}c^3\beta^3 - k\beta^3 + \beta^4\mathbb{Z}[\beta]\Big).$$

Since $\beta < cb+1 \le b^2$, we have

$$1 + c\beta' + (\beta')^3 = (\beta')^2/b + (\beta')^3 = (\beta')^2\,\frac{\beta - b^2}{b\beta} < 0.$$

This implies that $\gamma(\beta) < 1$ if $\binom{k+2}{3}c^3 + k$ is not a multiple of b.

It remains to consider the case that $\binom{k+2}{3}c^3 + k \equiv 0 \bmod b$, i.e.,

$$k \equiv -\frac{bk(k+2)}{6}c^2 k \bmod b.$$

Multiplying by $b-c$ gives

$$c \equiv -\frac{bk(k+2)}{6}c^3 \bmod b.$$

Note that $\frac{bk(k+2)}{6} = (b-c)\binom{k+2}{3} \in \mathbb{Z}$. We dinstinguish four cases:

- If $6 \perp b$, then $c \equiv 0 \bmod b$, contradicting that $1 \le c < b$.
- If $2 \mid b$ and $3 \nmid b$, then c is a multiple of $b/2$, i.e., $c = b/2$, $k = 1$. As k is also a multiple of $b/2$, we get that $b = 2$, thus $c = 1$. For $\beta^2 = 2\beta + 2$, we already know that $\gamma(\beta) < 1$.
- If $3 \mid b$ and $2 \nmid b$, then c and k are multiples of $b/3$. Since $k \in \mathbb{Z}$, we must have $c = 2b/3$, i.e., $k = 2$. As $2 \nmid b$, we obtain that $b = 3$, thus $c = 2$, and $\binom{k+2}{3}c^3 + k \not\equiv 0 \bmod b$.
- If $6 \mid b$, then c and k are multiples of $b/6$, thus $c \in \{b/2, 2b/3, 5b/6\}$, $k \in \{1,2,5\}$. If $k = 1$, then $b = 6$, thus $c = 3$, and $\binom{k+2}{3}c^3 + k \not\equiv 0 \bmod b$. If $k = 2$, then $b \in \{6,12\}$; we have excluded that $b = 6$, $c = 4$; for $b = 12$, $c = 8$, we have $\binom{k+2}{3}c^3 + k \not\equiv 0 \bmod b$. If $k = 5$, then $b \in \{6,30\}$; we have excluded that $b = 6$, $c = 5$; for $b = 30$, $c = 24$, we have $\binom{k+2}{3}c^3 + k \not\equiv 0 \bmod b$. □

Proof of Theorem 4.1. The case $\gamma(\beta)$ is treated in Propositions 4.4, 4.6 and 4.5. The case $a = b \ge 3$ in Corollary 3.3, where we prove that $\gamma(\beta) = 0$.

We can compute $\gamma(\beta)$ with arbitrary precision will the help of Theorem 3.1. All three values mentioned in this theorem are estimated using Proposition 3.5, where elements of $\mathcal{L}_y$ are enumerated using Proposition 3.6. □

Example 4.7. As an example, we will show the computation of $\gamma(\beta)$ for $\beta = 1+\sqrt{3}$, the Pisot root of $\beta^2 = 2\beta + 2$. Since b divides a, we know that we can choose every even digit and the even digit is then given uniquely. This allows us to consider shorter intervals than the ones in Lemma 3.4, namely, $[1+P'(w)+(\beta')^{2n+1}\frac{b-1}{1-(\beta')^2}, 1+P'(w)]$ for a prefix w of the length $2n$.

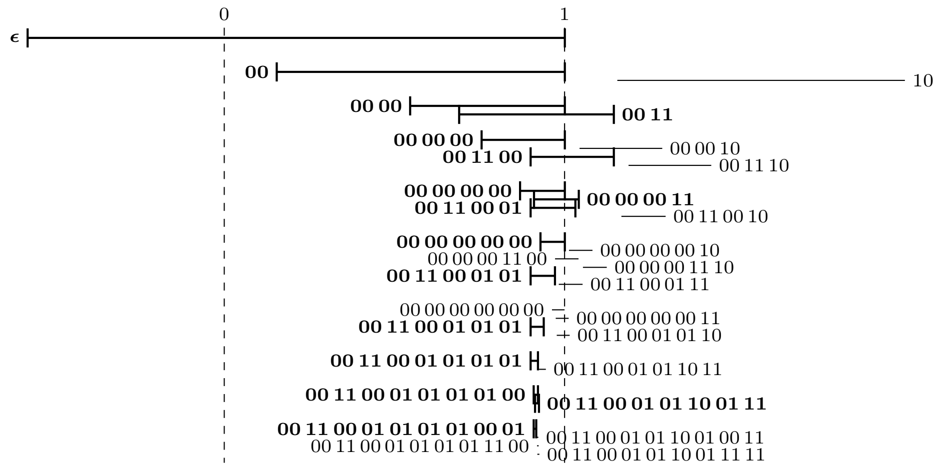

FIGURE 2. The computation of $\gamma(1+\sqrt{3})$. By a thick line we denote the intervals that we keep, by a thin line the ones that we 'forget'.

The computation is shown in Figure 2. We start with the interval for the empty word, which is $\left[1-\frac{\beta'(b-1)}{1-(\beta')^2}, 1\right]$. We then take the two values $0, 1$ for the digit x_0; the digit x_1 is fixed by this and we get the two prefixes $00, 10 \in \mathcal{L}_0$. However, the interval for 10 does not overlap the left-most interval (the one for 00 in this case), therefore we can 'forget' it. In each step, we then extend the length of the prefixes by two and we 'forget' the intervals that do not overlap the left-most one. The value of $\gamma(\beta)$ lies in the left-most interval. Already in the 5th step we obtain that $\gamma(\beta) \in [0.922, 0.971]$ therefore it is strictly between 0 and 1.

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