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Moebiovské systémy numerace s diskrétními grupami

Moebius numeration systems with discrete groups

DIPLOMOVÁ PRÁCE

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ZADÁNÍ DIPLOMOVÉ PRÁCE

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Obor:	Matematické inženýrství
Zaměření:	Matematické modelování
Název práce:	Moebiovské systémy numerace s diskrétními grupami / Moebius numeration systems with discrete groups

Osnova:

1. Studujte moebiovské systémy numerace, jejichž transformace generují diskrétní grupy.
2. Řešte otázku existence diskrétní grupy reálných moebiovských transformací s kompaktní fundamentální doménou a s celočíselnými koeficienty příslušných transformací.

Doporučená literatura:

1. P. Kůrka a A. Kazda. Moebius number systems based on interval covers. Nonlinearity 23(2010) 1031–1046.
2. A. F. Beardon. The geometry of dicrete groups. Springer-Verlag, Berlin 1995.
3. S. Katok. Fuchsian groups. The University of Chicago Press 1992.

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Vedoucí katedry Děkan

Prohlášení

Prohlašuji, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškerou použitou literaturu.

V Praze dne 16. dubna 2012 Tomáš Hejda

Název práce: **Möbiovské systémy numerace s diskrétními grupami**
Autor: Tomáš Hejda
Obor: Matematické inženýrství
Zaměření: Matematické modelování
Druh práce: Diplomová práce
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Abstrakt:

Moebiovské numerační systémy představují velmi obecnou konstrukci numeračních systémů zahrnující nejvýznamnější z nich — poziční systémy a řetězové zlomky. Tato konstrukce je výhodná především proto, že umožňuje dobrou geometrickou interpretaci a současně je algebraická — numerační systém je vždy podmnožina konečně generované grupy. V této práci se zabýváme existencí numeračních systémů, jejichž grupa je diskrétní a je generována racionálními Moebiovskými transformacemi, tedy lineárními či lineárně lomenými funkcemi s racionálními koeficienty.

Klíčová slova: numerační systémy, Fuchsovy grupy, Moebiovské transformace, racionální Moebiovské transformace

Title: **Möbius numeration systems with discrete groups**
Author: Tomáš Hejda

Abstract:

The theory of Möbius numeration systems brings a general construction of numeration systems, including the most studied ones—positional numeration systems and continued fractions. The advantages of this construction are a good geometrical exposition and an algebraic approach—numeration system is a subset of a finitely-generated group. This thesis investigates the existence of numeration systems such that the corresponding group is discreet and is generated by rational Möbius transformations, i.e. linear or linear-fractional functions with rational coefficients.

Keywords: numeration systems, Fuchsian groups, Möbius transformations, rational Möbius transformations

s velkým poděkováním mému trpělivému školiteli

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Remark. Even though we do most of the computations in the upper half-plane model $\mathbb{U}$, we draw most of the figures in the disc model $\mathbb{D}$ since in this model the hyperbolic space is represented by a bounded area in the complex plane. For details, see Remark 3.1.

CHAPTER 1

Introduction

This thesis treats some aspects of the Möbius number systems. Möbius number systems provide a general view on different kinds of numeration systems; in general any system based on linear and linear fractional transformations with the positive derivative. These include positional numeration systems, for instance the decimal and binary systems, in both redundant and non-redundant variants, but as well the Rényi systems [Rén57] with non-integer base. Continued fractions, when using the sign "−" in the fractions instead of "+", can be considered as a Möbius number system as well. Various properties and examples of Möbius number systems are given in [Kůr08, Kůr09b, KK10, Kůr12, Kůr11]. In the thesis, we are concerned about systems such that their transformations form a group.

In Chapter 2, we introduce the theory of Möbius number systems.

We view groups of transformations as groups of homeomorphisms of the hyperbolic plane, where the hyperbolic plane is represented as the upper complex half-plane; this is the contents of Chapter 3.

Fuchsian groups are discrete groups of transformations. Chapter 4 comprises the overview of these groups, some examples and some new results.

In Chapter 5, we are concerned with the groups of transformations with rational coefficients, we conjecture that no discrete groups of such transformations with an additional restriction on a boundedness of its fundamental domain exists and we explain the complications of the proof.

Chapter 6 contains the conclusions of the thesis.

The historical notes at the chapters' titles are taken from [wiki1, wiki2, wiki3, wiki4, wiki6, wiki7, www1].

CHAPTER 2

Möbius number systems

August Ferdinand Möbius (1790–1868)
German mathematician and theoretical astronomer, introduced the concept of homogeneous coordinates

We introduce the theory of Möbius number systems as is it introduced and studied in [Kůr08, Kůr09b, KK10]. To do so, we need some notions from the theory of combinatorics on words.

2.1. Combinatorics on words

Let $\mathcal{A}$ denote a finite *alphabet* with $\#\mathcal{A} \geq 2$, let $\mathcal{A}^+ := \bigcup_{n \geq 1} \mathcal{A}^n$ be the set of *finite non-empty words* over the alphabet $\mathcal{A}$. If we add the *empty word* λ, we get the set of all finite words over alphabet $\mathcal{A}$ and we denote it by $\mathcal{A}^*$. For a finite word $u = u_0 \dots u_{n-1} \in \mathcal{A}^*$ we denote $|u| := n$ its *length*. The Cantor space of *infinite words* is denoted by $\mathcal{A}^{\mathbb{N}}$ and is equipped with the metric $d(\boldsymbol{u}, \boldsymbol{v}) = 2^{-k}$ with $k \in \mathbb{N}$ being the first position where the words $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}^{\mathbb{N}}$ differ.

The set of words is naturally equipped with the operation of *concatenation* of words, which can be naturally extended to concatenating the sets of words, for instance, $UV = \{uv \mid u \in U, v \in V\}$ for arbitrary set of finite words U and set of finite or infinite words V.

A word $f \in \mathcal{A}^*$ is called a *factor* of a word $u \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, if there exists words x, y such that $u = xfy$, we denote this relation $f \sqsubset u$. A factor is called *prefix* if $x = \lambda$ and *suffix* if $y = \lambda$. All factors are considered finite. The set of all the factors of u is called *language* and is denoted by $\mathcal{L}(u)$. The language of any set of words is the union of languages of each of the words.

If $\boldsymbol{u} = x\boldsymbol{v}$ for some $x \in \mathcal{A}^*$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}^{\mathbb{N}}$, then $\boldsymbol{v}$ is called an *infinite suffix* or *tail* of the word $\boldsymbol{u}$.

A morphism $\varphi : \mathcal{A}^* \to \mathcal{B}^*$ is a map such that $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \mathcal{A}^*$. This means that a morphism is given by the images of the letters of $\mathcal{A}$. Any such morphism can be naturally extended to a map $\mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ putting $\varphi(u_0 u_1 u_2 \dots) := \varphi(u_0)\varphi(u_1)\varphi(u_2)\dots$. A morphism is called *substitution* if $\varphi(a) \neq \lambda$ for all $a \in \mathcal{A}$. Every substitution is continuous.

A *cylinder* of a finite word $u \in \mathcal{A}^*$ is a set of infinite words with a prefix u:

$$[u] := u\mathcal{A}^{\mathbb{N}} := \left\{ u\boldsymbol{v} \;\middle|\; \boldsymbol{v} \in \mathcal{A}^{\mathbb{N}} \right\}.$$

The cylinder is a *clopen set*, i.e., both open and closed, with respect to the metric d, which means that any finite union or intersection of cylinders is clopen, as well as the image of a cylinder under any substitution.

We define a *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ as $\sigma(u_0 u_1 u_2 u_3 \dots) = u_1 u_2 u_3 \dots$ We say that a set $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$ is a *subshift* if Σ is closed and σ-invariant. The whole set $\mathcal{A}^{\mathbb{N}}$ is a subshift and it is called *full shift*.

A subshift Σ is called *subshift of finite type* (abbreviated *SFT*), if there exists a finite set $X \subset \mathcal{A}^*$ of *forbidden words* such that

$$\Sigma = \left\{ \boldsymbol{u} \in \mathcal{A}^{\mathbb{N}} \,\middle|\, \mathcal{L}(\boldsymbol{u}) \cap X = \emptyset \right\}.$$

Every finite set X defines a subshift Σ_X by this equality.

More details on combinatorics on words can be found for instance in [Lot02].

Möbius number systems 2.2.

Definition 2.1. An *orientation-preserving real Möbius transformation* $M : \mathbb{C} \to \mathbb{C}$ is any map of the form

$$M(z) \coloneqq \frac{az+b}{cz+d}, \tag{2.1}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

The set of all orientation-preserving real Möbius transformations is denoted by $\mathfrak{M}(2, \mathbb{R})$. A transformation given by parameters a, b, c, d will be denoted $M_{a,b,c,d}$.

Later, in Section 3.2, we will see that $\mathfrak{M}(2, \mathbb{R})$ is a group and we will study more properties of Möbius transformations.

Definition 2.2. Let $F : \mathcal{A} \to \mathfrak{M}(2, \mathbb{R})$ be a system of orientation-preserving Möbius transformations $F_u : \mathbb{C} \to \mathbb{C}$ such that $F_{uv} = F_u \circ F_v$, which is given by generating MTs F_a, $a \in \mathcal{A}$. We say that F is a *Möbius iterative system*.

The *convergence space* $\mathbb{X}_F \subseteq \mathcal{A}^{\mathbb{N}}$ and the *symbolic representation* $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ are defined as

$$\begin{aligned}
\mathbb{X}_F &\coloneqq \left\{ \boldsymbol{u} \in \mathcal{A}^{\mathbb{N}} \,\middle|\, \lim_{n\to\infty} F_{\boldsymbol{u}_{[0,n)}}(i) \in \overline{\mathbb{R}} \right\}, \\
\Phi(\boldsymbol{u}) &\coloneqq \lim_{n\to\infty} F_{\boldsymbol{u}_{[0,n)}}(i),
\end{aligned}$$

where $i \in \mathbb{C}$ is the imaginary unit.

Let $\Sigma \subseteq \mathbb{X}_F$ be a subshift. We say that a pair (F, Σ) is a *Möbius number system* if $\Phi(\Sigma) = \overline{\mathbb{R}}$ and Φ is continuous. It is said to be *redundant* if for every continuous map $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $\Phi f = g\Phi$.

Definition 2.3. Let G be a group generated by the elements $\langle g_1, g_2, \dots, g_k\rangle$. For any element $h \in G$ we define its *length* to be the length of the shortest word $w = w_0 w_1 \cdots w_{n-1}$ over the $(2k)$-letter alphabet $\mathcal{A} := \{1, 2, \dots, k\} \cup \{-1, -2, \dots, -k\}$ such that $h = g_{w_0} g_{w_1} \cdots g_{w_{n-1}}$; we put $g_{-k} := g_k^{-1}$. The length is denoted by $\mathrm{len}_{g_1, g_2, \dots, g_k}(h)$.

Example 2.4 (Rényi positional system). Let us fix $\beta \in \mathbb{R}$, $\beta > 1$, denote $b := \lceil\beta\rceil - 1$. Let $\mathcal{A} = \{-b, -b+1, \dots, -1, 0, 1, \dots, b-1, b\} \cup \{\sharp\}$. Let $F_a(z) := \frac{z+a}{\beta}$ for $a \neq \sharp$, and let $F_\sharp(z) := \beta z$ be the generating MTs for a Möbius iterative system F. Let

$$\Sigma := \{\sharp^{\mathbb{N}}, 0^{\mathbb{N}}\} \cup (\sharp^* \cup 0^*)\big(\{-b, \dots, -1\}\{-b, \dots, 0\}^{\mathbb{N}} \cup \{1, \dots, b\}\{0, \dots, b\}^{\mathbb{N}}\big),$$

i.e. Σ contains a word $\sharp^{\mathbb{N}}$ and then words comprising either only non-positive or non-negative digits, and with any finite number of $\sharp$'s in front of the word.

This Σ is an SFT with forbidden words

$$X = \{\sharp 0\} \cup \big\{a\sharp \;\big|\; a \in \{-b, \dots, b\}\big\} \cup \big\{aa' \;\big|\; a, a' \in \{-b, \dots, b\}, a \cdot a' < 0\big\}.$$

All words $\boldsymbol{u} \in \Sigma$ have the form $\boldsymbol{u} = \sharp^k a_1 a_2 a_3 \cdots$ or $\boldsymbol{u} = \sharp^{\mathbb{N}}$. The symbolic representation has then a prescription

$$\Phi(\sharp^k a_1 a_2 a_3 \cdots) = \beta^k \sum_{j=1}^{\infty} \frac{a_j}{\beta^j} \quad \text{and} \quad \Phi(\sharp^{\mathbb{N}}) = \infty.$$

This system is in correspondence with Rényi expansions [Rén57]. Rényi defined the positional numeration systems on the interval $[0, 1)$ with the alphabet $\{0, 1, \dots, \lceil\beta\rceil - 1\}$ and proved that every $x \in [0, 1)$ has a representation. Using the negative digits, we extend the domain to $(-1, 1)$, and using the transformation $F_\sharp : z \mapsto \beta z$ arbitrarily many times allows representation of all real numbers. The word $\sharp^{\mathbb{N}}$ represents ∞. This means that (F, Σ) is a Möbius number system.

Example 2.5 (Continued fractions). One of the representations of real numbers is using *simple continued fractions.* Every $x \in \mathbb{R}$ can be expressed in the form of a finite or infinite fraction of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

with $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N} \setminus \{0\}$ for $k \geq 1$. An irrational number is expressed by a uniquie simple continued fraction, a rational number can be expressed by exactly two finite simple continued fractions.

We can modify this definition to obtain *alternating continued fractions* of the form

$$x = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots}}} \tag{2.2}$$

with $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{Z} \setminus \{0\}$ and $a_{k-1}a_k \leq 0$ for $k \geq 1$.

Let $\mathcal{A} := \{\overline{1}, 0, 1\}$ and put $F_0(z) := -1/z$, $F_1(z) := z + 1$ and $F_{\overline{1}}(z) := z - 1 = F_1^{-1}(z)$. Then x with a continued fraction (2.2) has a representation $1^{a_0}01^{a_1}01^{a_2}01^{a_3}0\cdots$, where we identify $1^{-j} = \overline{1}^j$.

Rational numbers have finite continued fractions, which have to be rewritten to infinite words. The fraction of the form

$$x = a_0 - \cfrac{1}{a_1 - \ddots \cfrac{1}{a_{n-2} - \cfrac{1}{a_{n-1}}}}$$

can be rewritten to

$$x = a_0 - \cfrac{1}{a_1 - \ddots \cfrac{1}{a_{n-2} - \cfrac{1}{a_{n-1} - \cfrac{1}{\pm\infty}}}},$$

hence x has a representation $1^{a_0}01^{a_1}0\cdots01^{a_{n-2}}01^{a_{n-1}}0b^{\mathbb{N}}$, where $b = 1$ for $a_{n-1} < 0$ and $b = \overline{1}$ for $a_{n-1} > 0$.

The set of all such representations is SFT Σ_X with the forbidden words $X = \{00, 1\overline{1}, \overline{1}1, 101, \overline{1}0\overline{1}\}$. The infinite words $1^{\mathbb{N}}$ and $\overline{1}^{\mathbb{N}}$ are not excluded and they both represent ∞.

We can see that the transformations of the system (F, Σ_X) form a group. For, we have $F_{00} = F_{1\overline{1}} = F_{\overline{1}1} = \mathrm{Id}$, $F_{101}(z) = \frac{z}{z+1} = F_{0\overline{1}0}(z)$ and $F_{\overline{1}0\overline{1}}(z) =$

$\frac{-z}{z-1} = F_{010}(z)$. We can summarize this by a list of rewriting rules, applying which we can convert any $u \in \mathcal{A}^*$ to some $u' \in \Sigma_X$ with $F_u = F_{u'}$:

$$00 \mapsto \lambda; \quad 1\overline{1} \mapsto \lambda; \quad \overline{1}1 \mapsto \lambda; \quad 101 \mapsto 0\overline{1}0; \quad \overline{1}0\overline{1} \mapsto 010.$$

Since (1) the first three rules diminish the length of the word; (2) the last two rules lower the number of occurences of 1 and $\overline{1}$; (3) the number of occurences of 0 is at most equal to one plus number of occurences of 1 and $\overline{1}$ (because 00 is forbidden); we get that the rewriting must terminate after finitely many steps.

Later in Chapter 4 we will see that this group is called *modular group*.

CHAPTER 3

Hyperbolic geometry

Jules Henri Poincaré (1854–1912)
French “polymath”, introduced two models of the hyperbolic geometry
that have been named in his honor

Euclid, in his treatise Elements, proposed 5 axioms of geometry, now being called Euclidean. The 5th axiom, so called "parallel postulate", says:

> "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Many mathematicians had thought that this axiom could have been omitted because it could be proven from the previous 4 ones. There were various attempts to prove the parallel postulate as well as to find its alternative. The discussion was ended by works of Lobachevsky, Gauss, Bolayi and Poincaré during the 19th century.

3.1. Hyperbolic models

Hyperbolic geometry is a non-Euclidean geometry such that through a given point, there exist more than one parallel line to every line. We use the Poincaré half-plane model and the Poincaré disc model. For a deeper study of the hyperbolic geometry, see [Bea95].

Denote $\mathbb{U} := \{z \in \mathbb{C} \mid \Im z > 0\}$ the *upper half plane*, and $\partial\mathbb{U} := \mathbb{R} \cup \{\infty\}$.

If we identify $\mathbb{U}$ with real pairs $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ putting $z = x + iy$, then the *hyperbolic metric* on the half-plane is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{3.1}$$

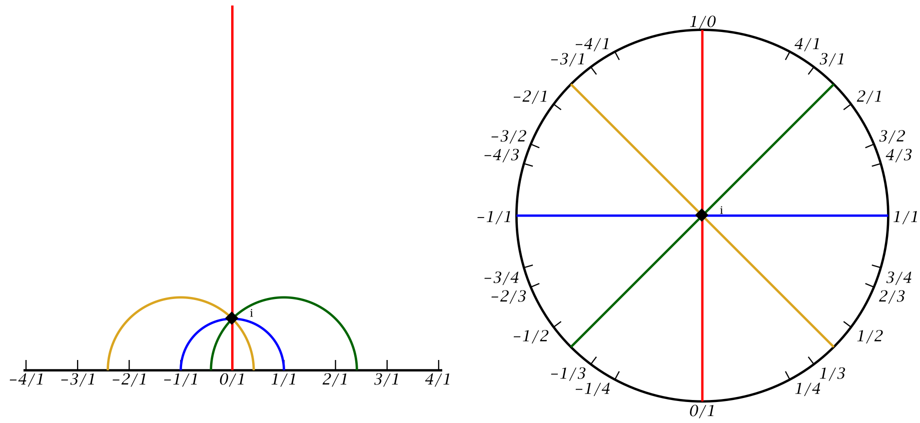
We define the *hyperbolic straight line* passing through points z_1, z_2 in the model as the *geodesic*, i.e. curve connecting these points such that it has the shortest length measured by (3.1). We denote such line $[z_1, z_2]$ and its hyperbolic length $\rho(z_1, z_2)$. It can be shown that these geodesics are parts of half-circles perpendicular to $\partial\mathbb{U}$ or half-lines starting on $\partial\mathbb{U}$ and perpendicular to $\partial\mathbb{U}$. We can extend the definition of a geodesic to the situation when z_1 or z_2 lies in $\partial\mathbb{U}$; in this case, the geodesic has an infinite hyperbolic length.

Denote $\mathbb{D} := \{z \in \mathbb{C} \mid |z| = 1\}$ the *unit disc* and $\partial\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ its boundary. Using map $\mathbf{d} : \mathbb{U} \cup \partial\mathbb{U} \to \mathbb{D} \cup \partial\mathbb{D}$,

$$\mathbf{d}(z) := \frac{iz + 1}{z + i}, \quad \mathbf{d}(\infty) = i, \tag{3.2}$$

we can define the disc model. The geodesics in this model are circular arcs perpendicular to $\partial\mathbb{D}$ and straight lines passing through the origin. The map (3.2) is *conformal*, i.e. it preserves angles between curves.

Figure 3.1. An arrangement in the hyperbolic geometry, drawn in the upper half-plane model (left) and the disc model (right). The drawing on the right illustrates the way how all the following figures in hyperbolic geometry will be drawn, cf. Remark 3.1.



Remark 3.1. We need to clarify the way we will draw all figures of objects in the hyperbolic geometry. The complication is that we identify the boundary $\partial\mathbb{U}$ of the geometry with real numbers $\overline{\mathbb{R}}$ using the upper half-plane model $\mathbb{U}$, but this model is unbounded so we cannot really draw it. Therefore we use the disc model $\mathbb{D} = \mathbf{d}(\mathbb{U})$, $\partial\mathbb{D} = \mathbf{d}(\partial\mathbb{U})$ for all the figures. However, we label all the points in the figures as points of the upper half-plane model, i.e. we label for instance the middle of the disc as i, since it is $0 = \mathbf{d}(i)$, see Figure 3.1 for an example.

The hyperbolic metric defines a topology on both $\mathbb{U}$ and $\mathbb{U} \cup \partial\mathbb{U}$.

Definition 3.2. Let $Q \subseteq \mathbb{U}$. We denote $\widetilde{Q}$ the closure of Q with respect to the topology on $\mathbb{U}$.

Let $Q \subseteq \mathbb{U} \cup \partial\mathbb{U}$. We denote $\overline{Q}$ the closure of Q with respect to the topology on $\mathbb{U} \cup \partial\mathbb{U}$.

3.2. Möbius transformations in hyperbolic geometry

Möbius transformations, as defined by (2.1), are conformal isometries of the hyperbolic plane $\mathbb{U}$. A map is *conformal* if it preserves the angles between curves. A map is an *isometry* of a metric space, if it is a bijection of this space and preserves the metric. This means that Möbius transformations map $\mathbb{U} \mapsto \mathbb{U}$ and we have already seen that they map $\partial\mathbb{U} \mapsto \partial\mathbb{U}$. Because of this, we will treat them as maps $\mathbb{U} \cup \partial\mathbb{U} \mapsto \mathbb{U} \cup \partial\mathbb{U}$.

Definition 3.3. The group $\mathrm{GL}^+(2,\mathbb{R})$ comprises all real 2×2 matrices with a strictly positive determinant.

Then we put

$$\mathrm{PGL}^+(2,\mathbb{R}) := \mathrm{GL}^+(2,\mathbb{R})/\sim,$$

where $\mathbf{A} \sim \mathbf{B}$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\mathbf{A} = \lambda\mathbf{B}$.

In the thesis we treat all matrices as elements of the group $\mathrm{PGL}^+(2,\mathbb{R})$, i.e. we identify a matrix $\mathbf{A}$ with a matrix $\lambda\mathbf{A}$ for arbitrary $\lambda \in \mathbb{R} \setminus \{0\}$.

To a Möbius transformation $M_{a,b,c,d}$ we assign a matrix $\mathbf{A}_M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}^+(2,\mathbb{R})$. This assignment is reasonable as matrices $\mathbf{A}$ and $\lambda\mathbf{A}$ for $\lambda \neq 0$ correspond to the same Möbius transformation, and the only real matrices that correspond to the same transformation as $\mathbf{A}$ are its multiples. We shall see that $M_{a,b,c,d}(M_{a',b',c',d'}(z)) = \frac{(aa'+bc')z+(ab'+bd')}{(ca'+dc')z+(cb'+dd')}$, which means that $\mathbf{A}_{MM'} = \mathbf{A}_M\mathbf{A}_{M'}$. The inverse matrix to $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix $\mathbf{A}^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ (we omit the factor $\frac{1}{\det \mathbf{A}}$ here because it does nothing in the sense of the group $\mathrm{PGL}^+(2,\mathbb{R})$). We see that a map $M \mapsto \mathbf{A}_M$ is an isomorphism $\big(\mathfrak{M}(2,\mathbb{R}), \circ\big) \mapsto \big(\mathrm{PGL}^+(2,\mathbb{R}), \cdot\big)$. Because $\mathrm{PGL}^+(2,\mathbb{R})$ is a group, we get that $\mathfrak{M}(2,\mathbb{R})$ is a group, called *Möbius group.*

In the disc model, the orientation-preserving Möbius transformation is given by

$$\widehat{M}(z) = \mathbf{d} \circ M \circ \mathbf{d}^{-1}(z).$$

The matrix of such orientation-preserving Möbius transformation is of the form

$$\mathbf{A}_{\widehat{M}} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha| > |\beta| \,.$$

We classify MTs according to their fixed points. We say that M is:

(1) *elliptic*—if M has a unique fixed point in $\mathbb{U}^\dagger$;

(2) *parabolic*—if M has a unique fixed point in $\partial\mathbb{U}$;

(3) *hyperbolic*—if M has exactly two fixed points in $\partial\mathbb{U}$.

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†Elliptic M, treated as a map $\mathbb{C} \mapsto \mathbb{C}$ instead of $\mathbb{U} \cup \partial\mathbb{U} \mapsto \mathbb{U} \cup \partial\mathbb{U}$, has another fixed point, that lies outside of $\mathbb{U}$.

It can be shown that there are no other cases. We denote s_M the unique fixed points of elliptic and parabolic transformation. Hyperbolic transformations have one stable fixed point s_M and one unstable fixed point u_M.

The *angle of rotation* of an elliptic M is the angle between the geodesics $[s_M, z]$ and $[s_M, M(z)]$ for arbitrary $z \in \mathbb{U}$, $z \neq s_M$. We denote this angle $\operatorname{rot} M$. The angle is signed (oriented) with the same orientation as in the complex plane.

A *trace of a matrix* is a sum of its diagonal elements. Based on this, we define *trace of an MT* as $\operatorname{tr}^2 M = \dfrac{\operatorname{tr}^2 \mathbf{A}_M}{\det \mathbf{A}_M} = \frac{(a+d)^2}{ad-bc}$. We define only the square of the trace, because we need the trace to be invariant under the map $\mathbf{A} \mapsto \lambda\mathbf{A}$ and this is the simplest and most common way to achieve this invariancy.

Using the trace, we can easily distinguish the classes od MTs by their matrices.

Proposition 3.4 ([Bea95, Theorem 4.3.4]). *Let M be an orientation-preserving Möbius transformation. Then M is:*

(1) *elliptic if and only if* $\operatorname{tr}^2 M < 4$*;*

(2) *parabolic if and only if* $\operatorname{tr}^2 M = 4$*;*

(3) *hyperbolic if and only if* $\operatorname{tr}^2 M > 4$.

The angle of rotation of an elliptic MT satisfies

$$\operatorname{tr}^2 M = 4\cos^2 \frac{\operatorname{rot} M}{2}.$$

We are concerned about so-called *rational Möbius transformations*, i.e. transformations with rational coefficients. We have already seen that $M_{\mathbf{A}} = M_{\lambda\mathbf{A}}$; the choice of λ equal to the product of the denominators of the coefficients shows that any rational Möbius transformation can be expressed as a transformation with integer coefficients. Since the inverse of a rational matrix is again rational, we get that rational orientation-preserving MTs form a subgroup of all orientation-preserving MT's denoted by $\mathfrak{M}(2, \mathbb{Z})$.

3.3. Expansion area and isometric circle

For defining expansion area and isometric circle, we will use the derivative of the Möbius transformation in the disc model, which is preferable due to its symmetries. For an MT $M = M_{\mathbf{A}}$, we define a *circle derivative* $M^{\bullet} : \mathbb{U} \cup \partial\mathbb{U} \to (0, \infty)$ as the modulus of the Euclidean derivative in the disc model:

$$M^{\bullet}(z) := \left|(\widehat{M})'(\mathbf{d}(z))\right| = \frac{\alpha\bar{\alpha} - \beta\bar{\beta}}{(\bar{\beta}\mathbf{d}(z) + \bar{\alpha})(\beta\overline{\mathbf{d}(z)} + \alpha)},$$

where $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \mathbf{A}_{\widehat{M}}$. For $z \in \overline{\mathbb{R}}$ we have

$$M^{\bullet}(z) = \frac{(ad - bc)(z^2 + 1)}{(az + b)^2 + (cz + d)^2}.$$

We define $M^{\bullet}(\infty)$ as a limit, which gives $M^{\bullet}(\infty) = \frac{ad - bc}{a^2 + c^2}$.

Using the circle derivative, we define the *isometric circle* as the area where $\widehat{M}$ acts as an Euclidean isometry:

$$I(M) := \{z \in \mathbb{U} \cup \partial\mathbb{U} \mid M^{\bullet}(z) = 1\}.$$

Moreover, we define the *expansion area*

$$\boldsymbol{V}(M) := \left\{z \in \mathbb{U} \cup \partial\mathbb{U} \;\middle|\; (M^{-1})^{\bullet}(z) > 1\right\}.$$

(Notice that the definition of $\boldsymbol{V}(M)$ uses the inverse of M.)

Proposition 3.5. **(1)** *For M that fixes i (i.e. $\widehat{M}$ fixes 0), we have $I(M) = \mathbb{U} \cup \partial\mathbb{U}$ and $\boldsymbol{V}(M) = \emptyset$.*

(2) *For any other M, the isometric circle is, when drawn in the disc model, an arc of a circle with the center c_M and radius r_M which can be computed from the matrix $\mathbf{A}_{\widehat{M}} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ as*

$$c_M = -\bar{\alpha}/\bar{\beta} \quad \text{and} \quad r_M = \sqrt{|c_M|^2 - 1}; \tag{3.3}$$

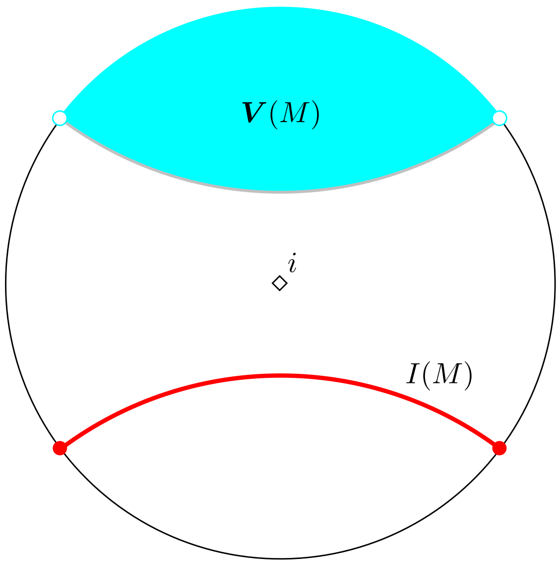
this circle is perpendicular to $\partial\mathbb{D}$ hence it is a geodesic. The expansion area $\boldsymbol{V}(M)$ is the interior of the isometric circle of M^{-1}.

Proof. **(1)** We have $0 = \frac{\alpha 0 + \beta}{\bar{\beta} 0 + \bar{\alpha}}$ if and only if $\beta = 0$. Then $\widehat{M}(z) = \frac{\alpha}{\bar{\alpha}} z$ and $\left|\widehat{M}(z)'\right| = 1$ for all $z \in \mathbb{C}$.

(2) We have $\widehat{M}(z)' = \frac{\alpha\bar{\alpha} - \beta\bar{\beta}}{(\bar{\beta}z + \bar{\alpha})^2}$, from whence it follows $\left|\widehat{M}(z)'\right| \gtreqless 1 \iff \left|\bar{\beta}z + \bar{\alpha}\right|^2 \lesseqgtr |\alpha|^2 - |\beta|^2 \iff \left|z - \left(-\frac{\bar{\alpha}}{\bar{\beta}}\right)\right|^2 \lesseqgtr \frac{|\alpha|^2}{|\beta|^2} - 1$. So the set where $\left|\widehat{M}(z)'\right| = 1$ is a circle with center $c_M = -\frac{\bar{\alpha}}{\bar{\beta}}$ and radius $r_M =$

The isometric circle in red and the expansion area in aquamarine for a transformation $M(z) := 4z$.

Figure 3.2.



$\sqrt{\frac{|\alpha|^2}{|\beta|^2} - 1} = \sqrt{|c_M|^2 - 1}$. The set where $\left|\widehat{M}^{-1}(z)'\right| > 1$ is then the interior of the isometric circle of M^{-1}.

The points 0 (center of $\partial\mathbb{D}$), c_M and intersection point of $x \in \partial\mathbb{D} \cap I(M)$ form a triangle that satisfies Pythagorean theorem and is therefore right-angled. Since any radius of a circle is perpendicular to the circle, we get that the circles are perpendicular at x. ***Q.E.D.***

An example of an isometric circle and an expansion area is given in Figure 3.2.

Proposition 3.6. *For every orientation-preserving Möbius transformation we have*

$$M\big(I(M)\big) = I(M^{-1}) \quad \textit{and} \quad M\big(\boldsymbol{V}(M^{-1})\big) = \big(\mathbb{U} \cup \partial\mathbb{U}\big) \setminus \big(\boldsymbol{V}(M) \cup I(M^{-1})\big).$$

Proof. The chain rule for derivative gives

$$1 = \mathrm{Id}'(z) = \widehat{M}\big(\widehat{M}^{-1}(z)\big)' = \widehat{M}'\big(\widehat{M}^{-1}(z)\big)(\widehat{M}^{-1})'(z). \tag{3.4}$$

This leads to $\widehat{M}'\big(\widehat{M}^{-1}(z)\big) = 1 \iff (M^{-1})'(z) = 1$, and according to this,

$$\begin{aligned} z \in I(M^{-1}) &\iff (M^{-1})'(z) = 1 \iff M'(M^{-1}(z)) = 1 \\ &\iff M^{-1}(z) \in I(M) \iff z \in M\big(I(M)\big). \end{aligned}$$

This proves the first claim.

To prove the second claim, consider the following equivalent steps that use (3.4), we consider $z \in \mathbb{U} \cup \partial\mathbb{U}$:

$$\begin{aligned} z \in M\big(\boldsymbol{V}(M^{-1})\big) &\iff M^{-1}(z) \in \boldsymbol{V}(M^{-1}) \iff M^{\bullet}\big(M^{-1}(z)\big) > 1 \\ &\iff (M^{-1})^{\bullet}(z) < 1 \iff z \notin I(M^{-1}) \cup \boldsymbol{V}(M). \end{aligned}$$

Q.E.D.

An interesting fact about the isometric circles is that $I(M)$ and $I(M^{-1})$ define the transformation M uniquely in the following sense.

Proposition 3.7. *Let $c_1, c_2 \in \mathbb{C}$ be two points such that $|c_1| = |c_2| > 1$. Then there exists exactly one orientation-preserving Möbius transformation M such that $c_1 = c_M$ and $c_2 = c_{M^{-1}}$.*

Proof. Let $\mathbf{A}_{\widehat{M}} = \left(\begin{smallmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right)$. Then $\mathbf{A}_{\widehat{M}^{-1}} = \left(\begin{smallmatrix} \bar\alpha & -\beta \\ -\bar\beta & \alpha \end{smallmatrix}\right)$. This means

$$c_1 = -\frac{\bar\alpha}{\bar\beta} \quad \text{and} \quad c_2 = \frac{\alpha}{\bar\beta}, \tag{3.5}$$

from which we get

$$\frac{c_2}{c_1} = -\frac{\alpha}{\bar\alpha} = e^{2i \arg \alpha},$$

which has a solution, because $\left|\frac{c_2}{c_1}\right| = 1$, and defines α up to a non-zero real multiple. Number β is then given by either equation in (3.5). ***Q.E.D.***

Proposition 3.8. *Let M be an orientation-preserving Möbius transformation not fixing i. Then M is:*

(1) *elliptic if and only if $I(M)$ and $I(M^{-1})$ intersect inside $\mathbb{U}$;*

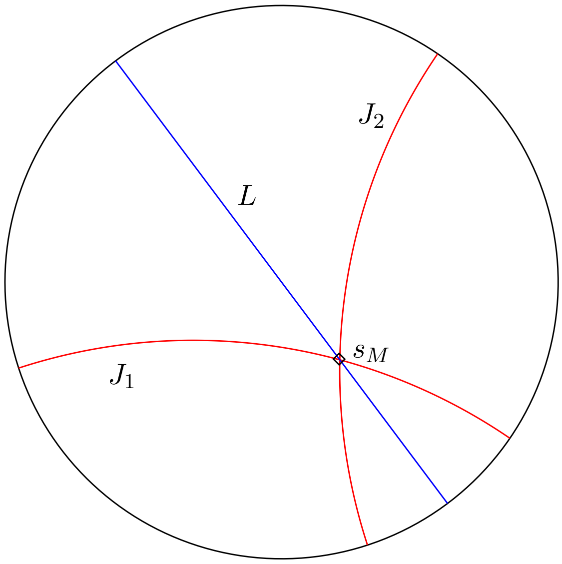
(2) *parabolic if and only if $I(M)$ and $I(M^{-1})$ intersect on $\partial\mathbb{U}$ (considering vertical parallel lines as intersecting in ∞);*

(3) *hyperbolic if and only if $I(M)$ and $I(M^{-1})$ are disjoint.*

Moreover, in the elliptic and parabolic case, $I(M) \cap I(M^{-1})$ is the fixed point of M.

For the proof, we need a notion of reflection in circle and line.

Isometric circles of M and M^{-1} and their line of symmetry in the disc model L—the elliptic case. **Figure 3.3.**



Definition 3.9 ([Bea95, §3.1.]). **(1)** Let S be an Euclidean circle $S = \{z \in \mathbb{C} \mid |z - c| = r\}$ with the center $c \in \mathbb{C}$ and radius $r > 0$. The *reflection in circle* S is then a map $\sigma_S : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$,

$$\sigma_S(z) \coloneqq c + \frac{r^2}{\overline{z - c}}, \quad \sigma_S(c) \coloneqq \infty, \quad \sigma_S(\infty) \coloneqq c.$$

(2) Let S be an Euclidean straight line $S = \left\{c + t\alpha \in \overline{\mathbb{C}} \,\middle|\, t \in \overline{\mathbb{R}}\right\}$ given by parameters $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and $c \in \mathbb{C}$. The *reflection in line* S is then a map $\sigma_S : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$,

$$\sigma_S(z) \coloneqq c + \alpha^2(\overline{z - c}), \quad \sigma_S(\infty) \coloneqq \infty.$$

The reflections in S have the following properties [Bea95, §3.1.–§3.2.]:

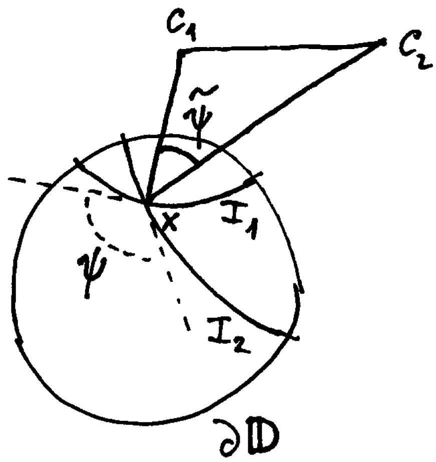
(1) $\sigma_S^2 = \mathrm{Id}$ *and* σ_S *fixes just and only the points in* S*.*

(2) *If* S *is a geodesic (in either half-plane or disc model), then* $\rho(z, S) = \rho(\sigma_S(z), S)$*, where as usual* $\rho(z, S) \coloneqq \inf_{w \in S} \rho(z, w)$ *and* ρ *is the hyperbolic distance (in half-plane or disc model, respectively).*

(3) *If* S *is a geodesic, then the geodesic* $[z, \sigma_S(z)]$ *is perpendicular to* S*.*

Proof of Proposition 3.8. We will use the Poincaré disc model for this proof. Let $J_1 \coloneqq \mathbf{d}(I(M))$ and $J_2 \coloneqq \mathbf{d}(I(M^{-1})) = \widehat{M}(J_1)$.

Figure 3.4. The angle ψ is the angle between two isometric circles with centers c_1, c_2.



The proof follows directly from [Kat92, Theorem 3.3.4] and [Bea95, §7.32.–7.34.]. In [Kat92], it is shown that any Möbius transformation is a reflection in its isometric circle followed by reflection in L—the *Euclidean* line of symmetry of J_1 and J_2 that passes through the point 0 (see Figure 3.3). Such L exists because $r_M = r_{M^{-1}}$ and $|c_M| = |c_{M^{-1}}|$. Because $\sigma_L(J_1) = J_2$ and σ_L fixes L, clearly $J_1 \cap L \subseteq J_1 \cap J_2$.

In [Bea95], it is shown that a transformation M is elliptic or parabolic or hyperbolic, when M is a composition of reflections in two geodesics that intersect in $\mathbb{U}$ or intersect on $\partial\mathbb{U}$ or are disjoint, respectively. ***Q.E.D.***

Remark 3.10. There is one special case in the Propositions 3.8 and 3.7, when the isometric circles of M and M^{-1} coincide. This happens only if $c_M = c_M^{-1}$, which means $c_M^2 = \mathrm{Id}$ and M is of order 2, hence it is elliptic.

Proposition 3.11. *Let M_1, M_2 be two orientation-preserving Möbius transformations not fixing i. Put*

$$\varrho = -\frac{|c_1|^2 + |c_2|^2 - |c_1 - c_2|^2 - 2}{2\sqrt{|c_1|^2 - 1}\sqrt{|c_2|^2 - 1}},$$

where $c_{1,2}$ are centers of the isometric circles given by (3.3). *Then the isometric circles $I(M_1)$ and $I(M_2)$ intersect if and only if $|\varrho| \leq 1$ and the angle ψ at the intersection point satisfies $\cos\psi = \varrho$.*

Proof. We will use the disc hyperbolic model. Suppose that the isometric circles $J_1 = \mathbf{d}(I(M_1))$ and $J_2 = \mathbf{d}(I(M_2))$ intersect, denote x their common point that lies in $\mathbb{D} \cup \partial\mathbb{D}$. The triangle $c_1 x c_2$ satisfies the (Euclidean) cosine rule $|c_1 - c_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\tilde{\psi}$, where $\tilde{\psi} = |\sphericalangle c_1 x c_2|$. Because $c_k x$ is perpendicular to J_k for $k = 1, 2$, we get $\psi = \pi - \tilde{\psi}$ (see Figure 3.4), whence

$$\cos\psi = -\cos\tilde{\psi} = -\frac{r_1^2 + r_2^2 - |c1 - c2|}{2r_1 r_2} = -\frac{|c_1|^2 + |c_2|^2 - |c_1 - c_2|^2 - 2}{2\sqrt{|c_1|^2 - 1}\sqrt{|c_2|^2 - 1}}.$$

Suppose that the isometric circles have no common point. This means $|c_1 - c_2| > r_1 + r_2$ and $|\varrho| > 1$ or we would be able to construct a triangle with side lengths $r_1, r_2, |c_1 - c_2|$—contradiction. ***Q.E.D.***

CHAPTER 4

Fuchsian groups

Lazarus Immanuel Fuchs (1833–1902)
German mathematician, contributed to the theory of differential equations and influenced Henri Poincaré

Definition 4.1. Let G be a group of orientation-preserving Möbius transformations. We say that G is a *Fuchsian group* if G acts discontinuously in $\mathbb{U}$.

A group G *acts discontinuously* in $\mathbb{U}$, if for every compact set $K \subset \mathbb{U}$ we have

$$K \cap M(K) = \emptyset$$

for all but finite number of $M \in G$.

"Acting discontinuously" is a topological property that can be studied for any group of homeomorphisms of a topological space.

Example 4.2. It is not easy to prove that a group of Möbius transformations is Fuchsian. In [Bea95, Example 9.4.4.] it is proved that the modular group is Fuchsian. The *modular group* is the group of transformations such that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The matrices of modular group are members of $\mathrm{PSL}(2, \mathbb{Z})$ defined as

$$\mathrm{PSL}(2,\mathbb{Z}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \;\middle|\; a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1 \right\} / \{I, -I\}.^{\dagger}$$

This group is generated by transformations $z \mapsto -1/z$ and $z \mapsto z + 1$. For the proof, it is very important to know the structure of the group.

When a group G is Fuchsian, it means that it is a discrete subgroup of the topological space of all Möbius transformations. Since a Fuchsian group G is discrete [Bea95, Theorem 8.4.1.], it is natural to ask for a geometrical interpretation of the quotient space $\mathbb{U}/G$. The following definition is adapted from [Bea95, Definition 9.1.1]; the authors there introduce a notion of "fundamental set", which we omit, therefore our definition is slightly modified.

Definition 4.3. Let G be a Fuchsian group. A subset $D \subseteq \mathbb{U}$ is a *fundamental domain* of the group G if

(1) D is a domain (open and connected);

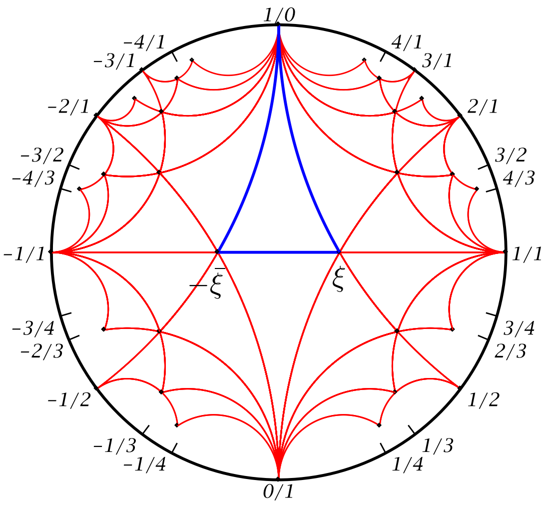
(2) $\bigcup_{M \in G} \widetilde{M(D)} = \mathbb{U}$ (recall that $\widetilde{\cdot}$ denotes the topological closure in $\mathbb{U} \cup \partial\mathbb{U}$, cf. Definition 3.2);

(3) $D \cap M(D) = \emptyset$ for all $M \in G \setminus \{\mathrm{Id}\}$;

(4) the boundary ∂D has zero hyperbolic area.

† Some authors identify modular group with $\mathrm{PSL}(2, \mathbb{Z})$, see [wiki5].

Figure 4.1. A fundamental domain and its images for the modular group.



It is obvious that D is a fundamental domain of G if and only if $M(D)$ is a fundamental domain of G for any $M \in G$; and D is a fundamental domain of G if and only if $M^{-1}(D)$ is a fundamental domain of $M^{-1}GM$ for any $M \in \mathfrak{M}(2, \mathbb{R})$.

Example 4.4. Let us take the modular group from Example 4.2. This group is Fuchsian and its fundamental domain is, for instance, the triangle with the vertices $\zeta, -\bar{\zeta}, \infty$, where $\zeta := e^{i\pi/3}$. On Figure 4.1, this fundamental domain is highlighted in blue, with its images in red.

Example 4.5. Another example of a Fuchsian group of rational transformations is a group shown on Figure 4.2 generated by two transformations

$$M_1 : z \mapsto 4z \qquad \text{and} \qquad M_2 := R^{-1} M_{22} R, \tag{4.1}$$

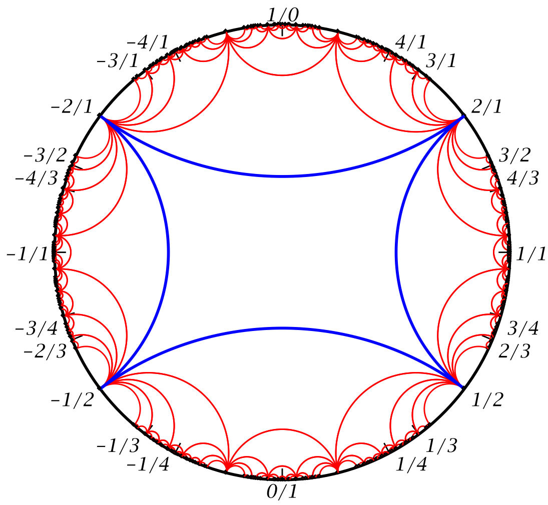
$$\text{where} \qquad R : z \mapsto \frac{z-1}{z+1} \qquad \text{and} \qquad M_{22} : z \mapsto 9z.$$

The proof that this group is Fuchsian will be given later in Proposition 4.13.

Both the above examples are Fuchsian groups that have 2 important properties:

- they consist only of rational Möbius transformations;

Figure 4.2. A fundamental domain and its images for the group from Example 4.5.



- their fundamental domain is unbounded in the hyperbolic plane.

In this thesis, we investigate the existence of a Fuchsian group of rational Möbius transformation that would have a bounded fundamental domain. The interest in bounded fundamental domain is given by the fact that unbounded domain denies redundancy of the corresponding Möbius number system.

4.1. Ford fundamental domains

Definition 4.6. Let H be a finite or countable set of orientation-preserving Möbius transformations such that no $M \in H$ fixes the point $i \in \mathbb{U}$. Then the set

$$P := \mathbb{U} \setminus \bigcup_{M \in H} \widetilde{\boldsymbol{V}(M)}$$

is called *pre-Ford domain* for the set H.

Theorem 4.7 ([Kat92, For25]). *Let G be a Fuchsian group such that i is not a fixed point of any elliptic $M \in G$. Then the pre-Ford domain for the set $H := G \setminus \{\mathrm{Id}\}$ is a fundamental domain for the group G.*

This theorem works only for the groups without elliptic transformations

M fixing i. It is clear that i is an interior point of P so $P \cap M(P) \neq \emptyset$ for M fixing i.

However, we can generalize the definition of the pre-Ford domain and obtain a similar result as the one in the previous theorem.

Definition 4.8. Let H be a set of Möbius transformation such that all the transformations in H fixing the point i form a cyclic group of finite order $r \geq 2$, this cyclic group is denoted by G_{el}. Let P be the pre-Ford domain for the set $H \setminus G_{\mathrm{el}}$. Then for arbitrary angle φ_0 the set

$$P \cap \mathbf{d}^{-1}\Big(\big\{Re^{i\varphi} \mid 0 < R < 1 \quad \text{and} \quad \varphi_0 < \varphi < \varphi_0 + \tfrac{2\pi}{r}\big\}\Big)$$

is called a *generalized pre-Ford domain* for the set H.

Theorem 4.9. *Let G be a Fuchsian group such that i is a fixed point of some elliptic $N \in G$. Then the generalized pre-Ford domain for the set G is a fundamental domain of G for arbitrary angle φ_0.*

Proof. This theorem can be proved in a similar way as shown in article [For25]. In Sections 2–5 of the article, the geometry and other aspects of members of the group are discussed. All these are valid for transformations $M \in G \setminus G_{\mathrm{el}}$. The Section 6 is crucial. They define the region P (denoted R there) and prove two facts that are sufficient for P to be a fundamental domain:

(1) no two interior points of P are congruent (i.e. are transformed one to the other by a member of the group);

(2) no region adjacent to P lying in $\mathbb{D}$ (denoted K there) can be added to P without the inclusion of points congruent to points in P.

We ought to adapt these two claims to our case:

(1′) no two interior points of P are congruent by a transformation $M \in G \setminus G_{\mathrm{el}}$;

(2′) no region adjacent to P lying in $\mathbb{D}$ can be added to P without the inclusion of points congruent to points in P.

The proof of the first property is as immediate as in the original case – any interior point of P is transformed by any transformation $M \in G \setminus G_{\mathrm{el}}$ to the interior of the isometric circle of M, hence outside P.

The proof of the second property is analogous to that in the article [For25]. Let us consider a point x on the boundary of P, on the isometric circle of M. Will will show that this point is carried on the boundary of P by M. The neighborhod of x is carried by M without the alternation of lengths, since $x \in I(M)$. Suppose $M(x)$ is not on ∂P. Since $M(x) \in I(M^{-1})$, it is not inside P, hence it is inside the isometric circle of some $M_1 \in G \setminus G_{\mathrm{el}}$. This means that the lengths in the neighborhood of x are magnified by $M_1 M$ and x lies inside its isometric circle—contradiction. Hence every x is carried by some $M \in G \setminus G_{\mathrm{el}}$ onto the boundary, and a region adjacent P in the neighborhood of x is transformed by M inside P. This proves the second property.

Now we know that images of P under transformations $M \in (G \setminus G_{\mathrm{el}}) \cup \{\mathrm{Id}\}$ cover the whole hyperbolic disc $\mathbb{D}$, i.e.

$$\mathbb{D} = \bigcup_{M \in (G \setminus G_{\mathrm{el}}) \cup \{\mathrm{Id}\}} M\big(\widetilde{P}\big). \tag{4.2}$$

Moreover $P \cap M(P) = \emptyset$ for $M \in G \setminus G_{\mathrm{el}}$. Let $N_0 \in G_{\mathrm{el}}$ be a transformation with minimal angle of rotation, which is equal to $2\pi/r$. Then $G_{\mathrm{el}} = \langle N_0 \rangle = \big\{N_0^k \mid k \in \{0, \dots, r-1\}\big\}$. Denote

$$Q := P \cap \mathbf{d}^{-1}\Big(\big\{re^{i\varphi} \mid 0 < r < 1 \quad \text{and} \quad \varphi_0 < \varphi < \varphi_0 + \tfrac{2\pi}{r}\big\}\Big).$$

Then

$$\widetilde{P} = \widetilde{\bigcup_{k=0}^{r-1} N_0^k(Q)} = \widetilde{\bigcup_{N \in G_{\mathrm{el}}} N(Q)}$$

and according to (4.2)

$$\mathbb{D} = \bigcup_{M \in G \setminus G_{\mathrm{el}} \cup \{\mathrm{Id}\}} M\big(\widetilde{P}\big) = \bigcup_{M \in G \setminus G_{\mathrm{el}} \cup \{\mathrm{Id}\}} \bigcup_{N \in G_{\mathrm{el}}} M\big(\widetilde{N(Q)}\big) = \bigcup_{M \in G} M\big(\widetilde{Q}\big).$$

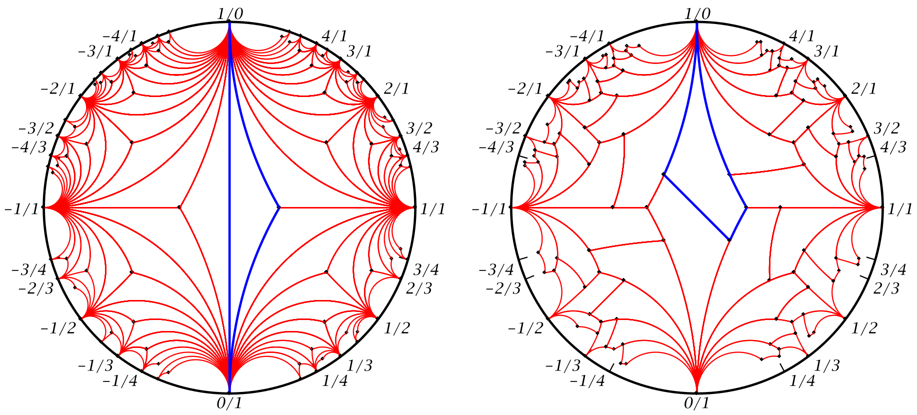
Q.E.D.

Definition 4.10. The fundamental domain from Theorem 4.7 is called *Ford fundamental domain.*

The fundamental domain from Theorem 4.9 is called *generalized Ford fundamental domain.*

Example 4.11. Consider again the modular group from Example 4.2. The transformation $z \mapsto -1/z$ of order 2 fixes i and is the only such elliptic transformation in the group. The set P for this group is a tetragon and the fundamental domain is its half. The shape of the domain depends on the choice of the angle φ_0. For the choice $\varphi_0 = 0$ see Figure 4.1. For the choices

Figure 4.3. Different generalized Ford fundamental domains for the modular group.



$\varphi_0 = -\pi/4$ and $\varphi_0 = -\pi/2$ see Figure 4.3. Notice that the number of sides of the fundamental domains differs with different choices of φ_0 (the notion of side will be exactly defined later in Definition 4.15).

Theorem 4.12. *Let $G = \langle M_1, \ldots, M_k\rangle$ be a Fuchsian group such that none of M_j fixes i and the regions $\boldsymbol{V}(M_1), \ldots, \boldsymbol{V}(M_k)$, $\boldsymbol{V}(M_1^{-1}), \ldots, \boldsymbol{V}(M_k^{-1})$ are pairwise disjoint. Then G is a Fuchsian group.*

Proof. Let us consider the Möbius iterative system with generating transformations $F_j := M_j$, $F_{k+j} := M_j^{-1}$ for $j \in \{1, \ldots, k\}$, with forbidden words $X := \big\{j(-j), (-j)j \;\big|\; j \in \{1, \ldots, k\}\big\}$; this means that the alphabet is $\mathcal{A} = \{-k, \ldots, -1\} \cup \{1, \ldots, k\}$ and the subshift Σ is a subshift of finite type. In this system, every member of G is representable, because $F_jF_{k+j} = F_{k+j}F_j = \mathrm{Id}$ for all j. Let us denote $Q_a := \mathbb{U} \smallsetminus \big(\boldsymbol{V}(F_a^{-1}) \cup I(F_a)\big) = \mathbb{U} \smallsetminus \boldsymbol{V}(F_a^{-1})$, for $a \in \mathcal{A}$.

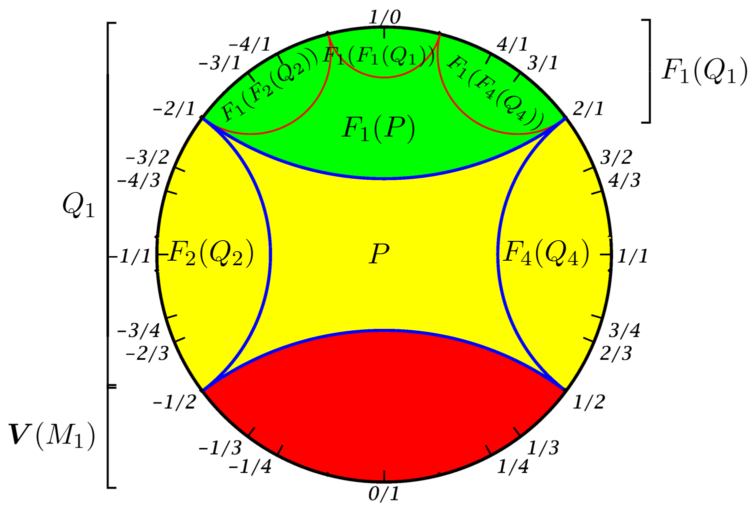
We will now show that for all $u \in \mathcal{L}(\Sigma)$ and $a, b \in \mathcal{A}$ such that $uab \in \mathcal{L}(\Sigma)$ we have

$$F_uF_bF_a(Q_a) \subseteq F_uF_b(Q_b).$$

The idea of the proof can be followed on Figure 4.4, which shows the situation for a group mentioned in Example 4.5.

(1) First, consider $u = \lambda$ and $ab \in \Sigma$. According to Proposition 3.6, we have $F_a(Q_a) = F_a\big(\mathbb{U} \smallsetminus \boldsymbol{V}(F_a^{-1})\big) = \mathbb{U} \smallsetminus F_a\big(\boldsymbol{V}(F_a^{-1})\big) = \mathbb{U} \smallsetminus \big(\mathbb{U} \smallsetminus$

Figure 4.4. The expansion area $\boldsymbol{V}(M_1)$ in red, the set Q_1 in yellow and green and the set $F_1(Q_1)$ in green.



$\boldsymbol{V}(F_a)) = \boldsymbol{V}(F_a)$. Since the expansion areas $\boldsymbol{V}(F_a)$ and $\boldsymbol{V}(F_b^{-1})$ are disjoint, because $ab \in \Sigma$, we have $F_a(Q_a) \subseteq Q_b$ and $F_bF_a(Q_a) \subseteq F_b(Q_b)$.

(2) Because F_u is a map, clearly $F_uF_bF_a(Q_a) \subseteq F_uF_b(Q_b)$ since we already have $F_bF_a(Q_a) \subseteq F_b(Q_b)$.

Now let us have $u = u_0u_1 \dots u_{n-1}$, then $F_u(Q_{u_{n-1}}) \subseteq F_{u_{[0,n-1)}}(Q_{u_{n-2}}) \subseteq \dots \subseteq F_{u_0u_1}(Q_{u_1}) \subseteq F_{u_0}(Q_{u_0})$.

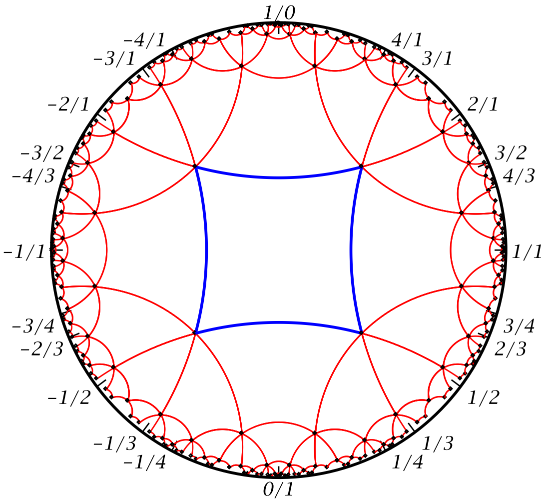
Let us consider a pre-Ford domain $P = \mathbb{D} \smallsetminus \bigcup_{a\in\mathcal{A}} \widetilde{\boldsymbol{V}(F_a)}$. We shall see that $P \subseteq Q_a$ and $P \cap F_a(Q_a) = \emptyset$ for all $a \in \mathcal{A}$. Because $F_u(P) \subseteq F_u(Q_{u_{n-1}}) \subseteq F_{u_0}(Q_{u_0})$ for any $u \in \mathcal{L}(\Sigma) \smallsetminus \{\lambda\}$, the images of P do not overlap.

According to Theorem 4.7, the Ford fundamental domain for this group is surely a subset of P, therefore it is clear that $\bigcup_{u\in\mathcal{L}(\Sigma)} F_u(P) = \mathbb{U}$.

Together, P is a fundamental domain for the group, which means that the group is Fuchsian. ***Q.E.D.***

Proposition 4.13. *The group $G = \langle M_1, M_2 \rangle$ given by* (4.1) *in Example 4.5 is Fuchsian.*

Figure 4.5. A pre-Ford domain for $\{M_1, M_2, M_1^{-1}, M_2^{-1}\}$ given by (4.3) and its images under elements of the group $\langle M_1, M_2 \rangle$ of the length up to 5 (cf. Remark 4.14).



Proof. The transformations $\widehat{F_1} := \widehat{M_1}$, $\widehat{F_2} := \widehat{M_2}$, $\widehat{F_3} := \widehat{M_1}^{-1}$ and $\widehat{F_4} := \widehat{M_2}^{-1}$ have matrices $\left(\begin{smallmatrix} 5 & 3i \\ -3i & 5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 & -4 \\ -4 & 5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 & -3i \\ 3i & 5 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 5 & 4 \\ 4 & 5 \end{smallmatrix}\right)$, respectively. This gives for the centers of isometric circles $c_1 = -5i/3$, $c_2 = 5/4$, $c_3 = 5i/3$ and $c_4 = -5/4$. The corresponding radii satisfy $r_1 = r_3 = 4/3$ and $r_2 = r_4 = 3/4$. It can be easily verified that $|c_a - c_b| \geq r_a + r_b$ for $a \neq b$, which is satisfactory for the expansion areas to be disjoint and the previous theorem can be used. ***Q.E.D.***

Remark 4.14. The condition on expansion areas to be pairwise disjoint is crucial for the proof of Theorem 4.12, because in that case the group contains no elliptic elements (for, the neighborhood of the fundamental domain P is tessellated by $F_a(P)$ for $a \in \mathcal{A}$, and none of them is elliptic or $\boldsymbol{V}(M)$ and $\boldsymbol{V}(M^{-1})$ intersect for any elliptic M not fixing i).

Consider for instance the group G generated by

$$\begin{aligned} & M_1 : z \mapsto qz && \text{for } q = \frac{\sqrt{3}+1}{\sqrt{3}-1} \\ \text{and} \quad & M_2 := R^{-1} M_1 R && \text{where } R : z \mapsto \frac{z-1}{z+1}. \end{aligned} \tag{4.3}$$

The pre-Ford domain for the set $H := \{M_1, M_2, M_1^{-1}, M_2^{-1}\}$ is a hyperbolic square with the angles at the vertices equal to $\pi/3$ (cf. Proposition 4.20).

However, the group element $N := M_1 M_2 M_1^{-1} M_2^{-1} M_1 M_2$ is elliptic of order 2 with a fixed point i, which means that P is not a fundamental domain for G, because $N(P) = P$. The domain P and its images under the elements of G of the length up to 5 are shown in Figure 4.5.

4.2. Sides of Ford fundamental domain

Since every isometric circle is a geodesic and the boundary of a (generalized) Ford fundamental domain consists of isometric circles and at most two geodesics at meeting the point i, the boundary of the domain is a union of geodesics.

Definition 4.15. Let P be a (generalized) pre-Ford domain for a set H satisfying $M \in H \Leftrightarrow M^{-1} \in H$. A *side* of a (generalized) pre-Ford fundamental domain P is any set $\ell \subset \mathbb{U}$ such that ℓ comprises more than one point and

$$\ell = \overline{P} \cap M\big(\overline{P}\big) \quad \text{for some} \quad M \in G \setminus \{\mathrm{Id}\}, \tag{4.4}$$

with the following exception: When $M(\ell) = \ell$, there's a fixed point s_M of M in the middle of $\ell = [z_1, z_2]$ (ℓ is necessarily a geodesic, see below). In this case, *sides* are $[z_1, s_M]$ and $[s_M, z_2]$ instead of the whole ℓ.

Since $P \cap M(P) = \emptyset$, ℓ is necessarily part of the boundaries of both sets. Because P is a convex set, the side is necessarily a geodesic. We call the endpoints of sides *vertices* of the domain P.

Let ℓ be a side and M be the corresponding group element. Then from (4.4) we have $M^{-1}(\ell) = M^{-1}\big(\overline{P}\big) \cap \overline{P}$, which means that $M^{-1}(\ell)$ is a side with M^{-1} being its side-pairing transformation.

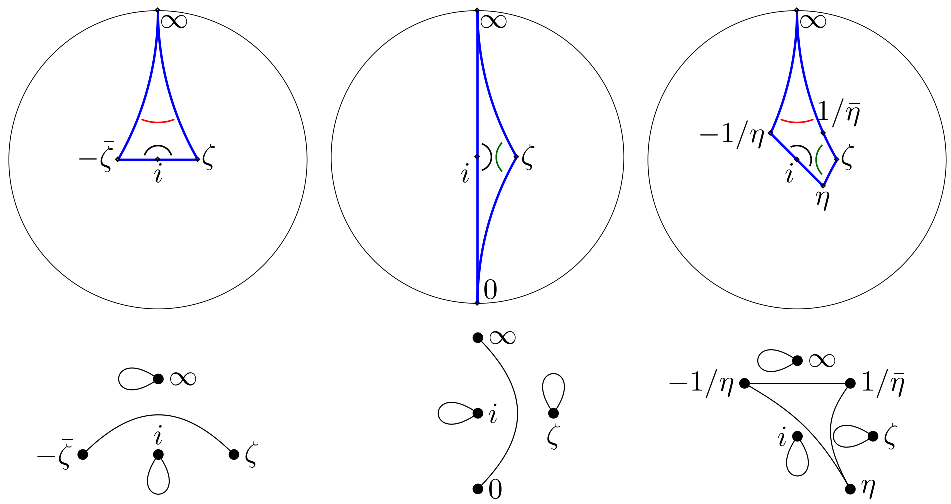
This justifies the following definition.

Definition 4.16. Let P be a (generalized) pre-Ford domain for the set H. For a side ℓ, let $M_\ell \in H$ be the transformation satisfying $\ell = \overline{P} \cap M\big(\overline{P}\big)$. The *side pairing* of the domain P for the set H is a mapping Π on the set of sides of P satisfying

$$\Pi(\ell) := M_\ell^{-1}(\ell).$$

Each point of a side ℓ that is not a vertex of P is mapped by M_ℓ^{-1} to a unique point in the side $\Pi(\ell)$. However, a vertex v belongs to exactly two sides of P, let us denote them ℓ_v and ℓ'_v. Let us now construct an

Figure 4.6. Side pairings of the different generalized Ford fundamental domains for the modular group—the black line corresponds to the transformation $z \mapsto -1/z$, the red line to $z \mapsto z+1$ and the green line to $z \mapsto (z-1)/z$ (top); the corresponding graphs $\mathcal{G}_P$ on the vertices of the domains (bottom) (cf. Example 4.17).



un-oriented graph[‡] $\mathcal{G}_P = (V, E)$ with V being the vertices of P and edges

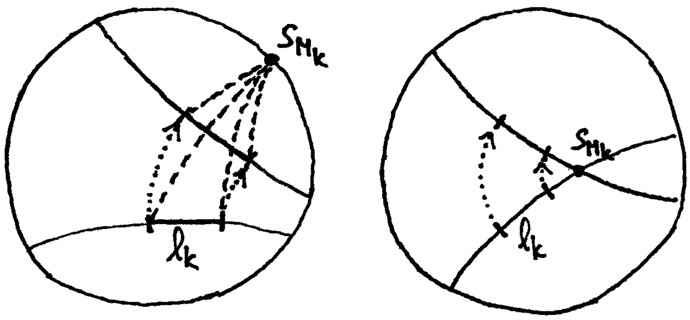
$$E := \left\{\{v, M_{\ell_v}^{-1}(v)\}, \{v, M_{\ell'_v}^{-1}(v)\} \middle| v \in V\right\}. \tag{4.5}$$

The connected components of V are either cycle graphs or simple complete graphs on two vertices or isolated vertices with loops. This is due to the fact that each vertex has an edge to only itself (if M_{ℓ_v} fixes v), to one vertex (if $M_{\ell_v}^{-1}(v) = M_{\ell'_v}^{-1}(v)$, in which case $M_{\ell_v}^{-1}(v)$ has edge only to v) or to two vertices (otherwise).

Example 4.17. A side of a domain need not to be a side in the common sense. In Figure 4.6, we can see that one geodesic comprises more sides

[‡] *Un-oriented graph* $\mathcal{G}$ is a pair of sets (V, E). The set V is arbitrary set and its elements are called *vertices*. The set E is a subset of $\binom{V}{2} \cup \binom{V}{1}$, where $\binom{V}{s}$ is a set of subsets of V of the size s; the elements of E are called *edges*. For $e \in E$, if we have $\#e = 2$ then $e = \{v, v'\}$ is an edge between vertices $v, v' \in V$; if we have $\#e = 1$ then $e = \{v\}$ is a loop on a vertex v. The set E can be interpreted as a relation "having common edge" on the set V—let us denote the symmetric, reflexive and transitive closure of this relation $\sim$. Then the connected component of V is a graph with $V' \subseteq V$ being an equivalence class of $\sim$ and $E' = E \cap \left(\binom{V'}{2} \cup \binom{V'}{1}\right)$. A graph (V, E) is a *cycle*, we write $V = (v_0, v_1, \dots v_{r-1})$, if $E = \left\{\{v_{k-1}, v_k\} \mid k \in \{1, \dots, r-1\}\right\} \cup \left\{\{v_0, v_1\}\right\}$. A graph (V, E) is a *simple complete graph* if $E = \binom{V}{2}$.

Figure 4.7. To the proof of Theorem 4.18, to illustrate the '+' sign in the sum of angles: the hyperbolic case in the left and the elliptic case in the right; the parabolic case is analogous to the elliptic one.



(two in the figure). The figure shows side-pairings for the three fundamental domains of the modular group, as they are depicted in Figures 4.1 and 4.3. Below each figure, the corresponging graphs $\mathcal{G}_P$ are drawn.

Theorem 4.18. *Let P be a (generalized) pre-Ford domain for a set of transformations H. Let $C = (v_0, v_2, \ldots v_{r-1})$, $r \geq 2$, be a cycle of a graph $\mathcal{G}_P$ given by (4.5). For $k \in \{0, \ldots r-1\}$, let $M_k \in G$ be the transformation mapping v_{k-1} to v_k and let α_k be the angle between the sides that meet at v_k where we consider the indices modulo r, i.e. $v_{-1} = v_{r-1}$ etc.*

Denote $\theta := \sum_{j=0}^{r-1} \alpha_j$ and $N := M_{r-1} M_{r-2} \cdots M_1 M_0$. Then:

(1) *N is identity if and only if $\theta \in 2\pi\mathbb{Z}$;*

(2) *N is elliptic with $\operatorname{rot} N = \theta$ if and only if $\theta \notin 2\pi\mathbb{Z}$.*

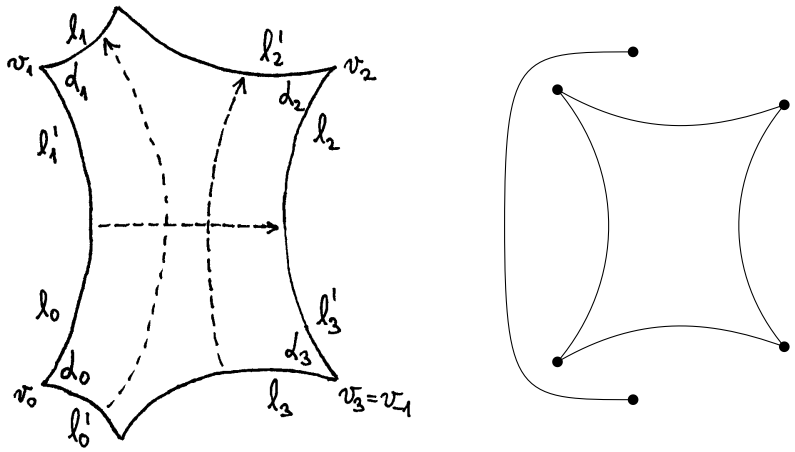
Proof. Let ℓ_j be the side of P such that $M_j^{-1}(\ell_j)$ is a side as well from which it is clear that the sides meeting at v_j are exactly ℓ_j and $\ell'_j := M_{j+1}^{-1}(\ell_{j+1})$. The angle between ℓ_0 and ℓ'_0 is α_0.

From the conformity of Möbius transformations we know that the angle between $M_0(\ell_0)$ and $M_0(\ell'_0) = \ell_1$ is α_0 as well. The angle between ℓ_1 and ℓ'_1 is α_1, hence the angle between $M_0(\ell_0)$ and ℓ_1 is $\alpha_0 + \alpha_1$.

Similarly the angle between $M_1(M_0(\ell_0))$ and ℓ_2 is $\alpha_0 + \alpha_1 + \alpha_2$. After r steps we get that the angle between $M_{r-1}(M_{r-2}(\cdots(M_1(M_0(\ell_0))))) = N(\ell_0)$ and $\ell_r = \ell_0$ is $\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1} = \theta$.

It remains to explain why the angle α_k is always added to the sum and never subtracted. For a hyperbolic transformation M_k, consider the triangle with vertices s_{M_k} (one of its fixed points), and the vertices of ℓ_k. Since the

Figure 4.8. An example of a domain with 6 sides and a cycle in the graph $\mathcal{G}_P$ of the length 4 (left) and the whole graph $\mathcal{G}_P$ (right) (cf. Theorem 4.18).



transformation is orientation-preserving, i.e. it preserves orientation of any hyperbolic triangles, the side ℓ_k is mapped as depicted in Figure 4.7. For a parabolic or elliptic transformation, consider the vertices of ℓ_j and the fixed point s_{M_k}. Since the transformation keeps the hyperbolic distance unaltered, it preserves the order of these points on the geodesic $I(M_k^{-1})$, as depicted in Figure 4.7.

Transformation N fixes the point v_0. If θ is a multiple of 2π, N is an orientation-preserving Möbius transformation that fixes every point on ℓ_0 which means that it is the identity. Otherwise it is an elliptic transformation with θ being its angle of rotation. ***Q.E.D.***

For an illustrative example, see Figure 4.8, where the situation is shown for a Ford fundamental domain consisting of 6 sides, with a cycle in the graph $\mathcal{G}_P$ of the length 4.

The theorem, in many cases, instantly shows the existence of elliptic element in a group. For instance, the graph $\mathcal{G}_P$ for the domain in Figure 4.5 (cf. Remark 4.14) is a cycle, all the angles are equal to $\pi/3$. This gives that for each vertex of the domain, there exist an elliptic transformation that fixes it and has the angle of rotation equal to $2\pi/3$, these group elements are $M_1M_2M_1^{-1}M_2^{-1}$ and its conjugates. Beardon [Bea95] does not discuss the necessity of existence of the elliptic transformations in a Fuchsian group with a bounded fundamental domain. We state the following conjecture, whose justification is explained below in Example 4.21.

Conjecture 4.19. There exist a Fuchsian group G with a bounded fundamental domain, such that G contains no elliptic transformations.

We will use the following statement from [Kůr09a].

Proposition 4.20. *Let $2m$, $2n$ be positive integers such that $1/m + 1/n < 1$. Put $\phi := \pi/n$, $\psi := \pi/m$ and $q := \left(1 + \sqrt{1 - \frac{\sin^2 \phi/2}{\cos^2 \psi/2}}\right) \Big/ \left(1 - \sqrt{1 - \frac{\sin^2 \phi/2}{\cos^2 \psi/2}}\right)$. Let us define the following transformations:*

- $F_0 : z \mapsto z/q$, *hence* $\hat{F}_0 : z \mapsto \frac{(q+1)z - i(q-1)}{i(q-1)z + (q+1)}$;
- $\widehat{R} : z \mapsto e^{i\phi} z$ *(rotation around* 0 *by the angle* ϕ *in the disc model), i.e.* $R : z \mapsto \frac{z \cos \phi/2 + \sin \phi/2}{-z \sin \phi/2 + \cos \phi/2}$;
- $F_k : z \mapsto R^k F_0 R^{-k}(z)$.

Then the set $P = \mathbb{U} \setminus \bigcup_{k=0}^{2n} \boldsymbol{V}(F_k)$ is a regular $2n$-gon whose inner angles are equal to ψ.

Proof. We first show that $q \in \mathbb{R}$ and $q > 1$—for this, see the following where each inequality implies the next one:

$$0 < 1/m + 1/n < 1;$$
$$\cos(\pi/2m + \pi/2n) > 0 \quad \text{and} \quad \cos(\pi/2m - \pi/2n) > 0;$$
$$2\cos(\pi/2m + \pi/2n)\cos(\pi/2m - \pi/2n) > 0;$$
$$\cos \pi/m + \cos \pi/n > 0;$$
$$2\cos^2 \pi/2m + 2\cos^2 \pi/2n - 2 > 0;$$
$$\cos^2 \pi/2m > \sin^2 \pi/2m;$$
$$\sqrt{1 - \frac{\sin^2 \phi/2}{\cos^2 \psi/2}} > 0;$$
$$q > 1.$$

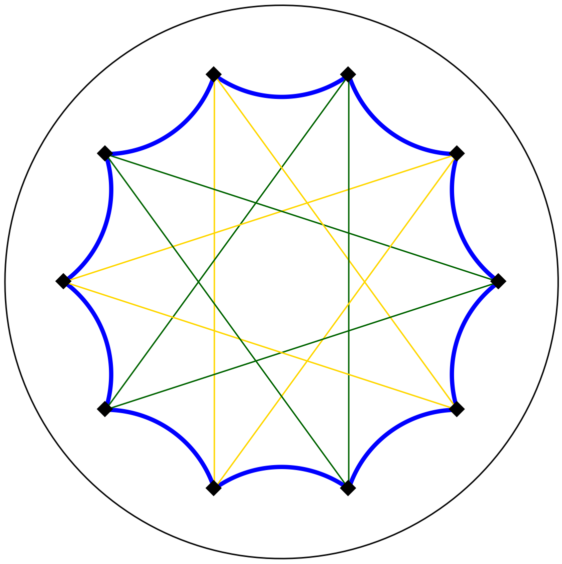
Since $R^{2n} = \mathrm{Id}$, we immediately see that the the set P satisfies a $2n$ rotational symmetry in the disc model. The value of inner angles can be computed using Proposition 3.11. For the simplicity, let us measure the angle between $I(F_0)$ and $I(F_1)$. Using (3.3), we see that

$$c_0 = i\frac{q+1}{q-1} \quad \text{and} \quad c_1 = c_0 e^{i\phi}. \tag{4.6}$$

Denote $c = |c_0| = |c_1|$. Then $|c_0 - c_1| = 2c \sin \phi/2$ and

$$-1 + \frac{c^2}{c^2 - 1}\sin^2 \phi/2 = -\frac{2c^2 - 4c^2 \sin^2 \phi/2 - 2}{2(c^2 - 1)} \overset{\text{Prop. 3.6}}{=\!=\!=} \cos\psi = \cos^2 \psi/2 - 1.$$

Figure 4.9. The pre-Ford domain P from Example 4.21 in blue and the cycles of graph $\mathcal{G}_P$ in green and yellow. The inner angles are all equal to $2\pi/5$.



Let us denote $\Phi = \sin^2 {}^{\phi}\!/_{2}$ and $\Psi = \cos^2 {}^{\psi}\!/_{2}$. Then we can express $1/c$ in the terms of Φ, Ψ:

$$
\begin{aligned}
-1 + \frac{c^2}{c^2-1}\Phi &= \Psi - 1; \\
\frac{c^2}{c^2-1} &= \Psi/\Phi; \\
c^2(1 - \Psi/\Phi) &= -\Psi/\Phi; \\
1/c^2 &= 1 - \Phi/\Psi.
\end{aligned}
$$

From (4.6) we see that $qc - c = q + 1$ and

$$q = \frac{c+1}{c-1} = \frac{1+1/c}{1-1/c} = \frac{1+\sqrt{1-\Phi/\Psi}}{1-\sqrt{1-\Phi/\Psi}}.$$ ***Q.E.D.***

Example 4.21. Let us apply the previous proposition to the case $n = 5$ and $m = 2.5$ and take the group $G := \langle F_0, F_1, \dots F_4\rangle$.

As we already said, proving that a group is Fuchsian is generally very difficult. Suppose now that G is Fuchsian and a pre-Ford domain for the set of transformations $\{F_0, \dots, F_4, F_0^{-1}, \dots, F_4^{-1}\}$ is its Ford fundamental domain.

This domain is, according to the previous proposition, a regular hyperbolic 10-gon with the angles at its vertices all equal to $2\pi/5$ and it is bounded. This domain embodies a side-pairing given by the generators F_k. The graph $\mathcal{G}_P$ comprises two cycles of the length 5. (In Figure 4.9, the sides of P are depicted in blue, its vertices in black, and the graph $\mathcal{G}_P$ in green and yellow.)

For Theorem 4.18, we have all α_k equal to $2\pi/5$ and $r = 5$, hence $\theta = 2\pi$. This means that the transformations satisfy $F_0F_2F_4F_1^{-1}F_3^{-1} = F_0F_3^{-1}F_1^{-1}F_4F_2 = \mathrm{Id}$, which is obviously true for all the conjugates of these two as well.

Under the assumption that P is a fundamental domain, we get that the group contains no elliptic transformations—if there existed an elliptic $M \in G$, its fixed point would have to be a vertex of an image of P, and by conjugation there exists another elliptic $M_1 \in G$ with the fixed point being a vertex of P itself, let us denote it v. Then $M_1(P)$ meets the neighborhood of v, but the neighborhood of v is tessellated by images of P under hyperbolic transformations, contradiction. This led us to state Conjecture 4.19.

CHAPTER 5

Rational groups with a bounded fundamental domain

Julius Wilhelm Richard Dedekind (1831–1916)
German mathematician, worked in algebra and the foundations of the real numbers, discovered the tessellation by the modular group.

Table 5.1. The significant values of the trace of a Möbius transformation.

$\mathrm{tr}^2 M$	$\mathrm{rot}\, M = 2 \arccos \frac{\sqrt{\mathrm{tr}^2 M}}{2}$	order of M
0	π	2
1	$2\pi/3$	3
2	$\pi/2$	4
3	$\pi/3$	6

Our concern is to study *rational groups*, i.e. groups of transformations with matrices in $\mathrm{PGL}^+(2, \mathbb{Z})$, which comprises matrices with integer coefficients and a positive determinant. Let us recall that we identify the matrices $\mathbf{A}$ and $\lambda \mathbf{A}$, which means that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \frac{1}{\det \mathbf{A}} \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$; for the same reason we speak about "rational groups" and consider the matrices to have integer coefficients.

In Examples 4.2 and 4.5 we show that there exist rational Fuchsian groups, but in both cases, their fundamental domain is unbounded.

We have not found any example of a rational Fuchsian group with bounded fundamental domain, so we stated the following conjecture.

Conjecture 5.1. There is no rational Fuchsian group with a bounded fundamental domain.

However, the proof of this conjecture seems to be unreachable. Even though, we try to investigate some cases.

5.1. Groups with elliptic elements

Suppose first that the group G contains elliptic elements. Since we consider Fuchsian groups, the angle of rotation of any elliptic $M \in G$ satisfy $\mathrm{rot}\, M \in \pi\mathbb{Q}$; or in the case $\mathrm{rot}\, M \notin \pi\mathbb{Q}$, the powers of M accumulate in $\mathfrak{M}(2, \mathbb{R})$. This puts a restriction on the values of the trace $\mathrm{tr}^2 M$.

Proposition 5.2. *Let G be a rational Fuchsian group and $M \in G$ be an elliptic transformation. Then*

$$\mathrm{tr}^2 M \in \{0, 1, 2, 3\}.$$

Proof. From Proposition 3.4 we know that $\mathrm{tr}^2 M = 4 \cos^2 \frac{\mathrm{rot}\, M}{2}$. Any angle θ satisfies $2 \cos^2 \theta = 1 + \cos 2\theta$. Hence $\mathrm{tr}^2 M = 2(1 + \cos \mathrm{rot}\, M)$. Because $\mathbf{A}_M$ has rational elements, $\mathrm{tr}^2 M \in \mathbb{Q}$, which follows to $\cos \mathrm{rot}\, M \in \mathbb{Q}$.

In [Olm45], it is shown that when $\theta \in \pi\mathbb{Q}$ satisfies $\cos\theta \in \mathbb{Q}$, then $\cos\theta \in \{0, \pm\frac{1}{2}, \pm 1\}$. Because $\operatorname{rot} M \in \pi\mathbb{Q}$ and $\cos\operatorname{rot} M \in \mathbb{Q}$, we have $\cos\operatorname{rot} M \in \{0, \pm\frac{1}{2}, \pm 1\}$ and

$$\operatorname{tr}^2 M \in 2\Big(1 + \big\{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\big\}\Big) = \{0, 1, 2, 3\}. \qquad \textbf{\textit{Q.E.D.}}$$

In Table 5.1, we have the significant values of trace from the previous theorem and the corresponding angles of rotation. The information in the table leads to the following corollary.

Theorem 5.3. *A rational Fuchsian group contains only elements of orders* $1, 2, 3, 4, 6, \infty$.

Proof. Hyperbolic and parabolic elements have infinite order, elliptic elements have orders $2, 3, 4, 6$ according to Table 5.1. The order 1 corresponds to the identity transformation. ***Q.E.D.***

We now try to transform the problem of existence of a rational Fuchsian group with bounded domain to a problem of solving a system od Diophantine conditions.

Let us suppose that in G, there exist two elliptic elements $M_{1,2}$ having different fixed points, such that the composition $M_1 \circ M_2$ is elliptic as well. Denote $\mathbf{A}_{M_1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\mathbf{A}_{M_2} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then $\mathbf{A}_{M_1 M_2} = \left(\begin{smallmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{smallmatrix}\right)$. According to Proposition 5.2, we have the equations

$$\operatorname{tr}^2 M_1 M_2 = \frac{(aA + bC + cB + dD)^2}{(ad - bc)(AD - BC)} \in \{0, 1, 2, 3\}; \tag{d1}$$

$$\operatorname{tr}^2 M_1 = \frac{(a + d)^2}{ad - bc} \in \{0, 1, 2, 3\}; \tag{d2}$$

$$\operatorname{tr}^2 M_2 = \frac{(A + D)^2}{AD - BC} \in \{0, 1, 2, 3\}. \tag{d3}$$

We emphasize the different fixed points, because when M_1 and M_2 have the same fixed point and $\operatorname{rot} M_{1,2} \in \pi\mathbb{Q}$, then clearly $M_1 M_2$ has the same fixed point and $\operatorname{rot} M_1 M_2 = \operatorname{rot} M_1 \pm \operatorname{rot} M_2 \in \pi\mathbb{Q}$. The fixed point s_1 of M_1 satisfies $s_1 = \frac{as_1+b}{cs_1+d}$, equivalently $cs_1^2 + (d - a)s_1 - b = 0$; the fixed point s_2 of M_2 satisfies $Cs_2^2 + (D - A)s_2 - B = 0$. Because all the coefficients are real, we shall see that $s_1 = s_2$ if and only if $(c, d - a, b) = \lambda(C, D - A, B)$ for some $\lambda \neq 0$. Enforcing $s_1 \neq s_2$ is then equivalent to satisfying one of the inequalities

$$c(A - D) - (a - d)C \neq 0; \tag{d4}$$

$$\text{or} \quad b(A - D) - (a - d)B \neq 0; \tag{d5}$$

let us denote the last two left-side terms ω_c and ω_b, respectively.

All these considerations can be summarized in the following way.

Theorem 5.4. *Suppose that the system of conditions* (d1)–(d5) *has no integer solutions. Then there exist no rational Fuchsian group G containing elliptic transformation $M_{1,2}$ with different fixed points such that $M_1 M_2$ is elliptic.*

Unfortunately, this theorem gives only a partial results, in 2 senses:

(1) we do not know whether the system has integer solutions;

(2) we do not know whether elliptic transformations have to be contained in a Fuchsian group with a bounded fundamental domain;

(3) we do not know whether when the group contains elliptic transformations, it contains an elliptic composition of elliptic transformations with different fixed points.

These three questions are essential for the Conjecture 5.1. The previous theorem would then answer the conjecture for a broad family of Fuchsian groups, but it would not answer it completely.

Using a computer simulation, we have shown the following.

Claim 5.5. *The system of conditions* (d1)–(d5) *has no integer solution with* $a, b, c, d, A, B, C, D \in \{-100, -99 \ldots, 99, 100\}$.

The description of the program can be found in Section 5.2. To make the program run significantly faster, we use following several symmetries of the conditions:

Lemma 5.6. *Let Sol be the set 8-tuplets* $(a, b, c, d, A, B, C, D) \in \mathbb{Z}^8$ *of the solutions of the system* (d1)–(d5). *Then*

$$\begin{aligned}
& v^{(1)} := (a, b, c, d, A, B, C, D) \in \mathit{Sol} \\
\iff & v^{(2)} := (A, B, C, D, a, b, c, d) \in \mathit{Sol} \\
\iff & v^{(3)} := (a, c, b, d, A, C, B, D) \in \mathit{Sol} \\
\iff & v^{(4)} := (d, b, c, a, D, B, C, A) \in \mathit{Sol} \\
\iff & v^{(5)} := (-a, b, c, -d, -A, B, C, -D) \in \mathit{Sol} \\
\iff & v^{(6)} := (a, -b, -c, d, A, -B, -C, D) \in \mathit{Sol}
\end{aligned}$$

Proof. Let us denote $f_i(a,b,c,d,A,B,C,D)$ the formula in the condition (di), for $i = 1, \ldots, 5$. Then clearly

$$\begin{aligned}
f_1(v^{(1)}) = f_1(v^{(2)}) = f_1(v^{(3)}) = f_1(v^{(4)}) = f_1(v^{(5)}) = f_1(v^{(6)}),\\
f_2(v^{(1)}) = f_3(v^{(2)}) = f_2(v^{(3)}) = f_2(v^{(4)}) = f_2(v^{(5)}) = f_2(v^{(6)}),\\
f_3(v^{(1)}) = f_2(v^{(2)}) = f_3(v^{(3)}) = f_3(v^{(4)}) = f_3(v^{(6)}) = f_3(v^{(6)}),\\
f_4(v^{(1)}) = -f_4(v^{(2)}) = f_5(v^{(3)}) = -f_4(v^{(4)}) = -f_4(v^{(5)}) = -f_4(v^{(6)}),\\
f_5(v^{(1)}) = -f_5(v^{(2)}) = f_4(v^{(3)}) = -f_5(v^{(4)}) = -f_5(v^{(5)}) = -f_5(v^{(6)}),
\end{aligned}$$

which is satisfactory for the equivalence of the solutions of the system. ***Q.E.D.***

Example 5.7. We now show a few examples of "partial solutions"—when we omit some of the conditions, we get a solvable Diophantine system.

- Let $\mathbf{A}_{M_1} = \left(\begin{smallmatrix} 6 & -3 \\ 7 & -3 \end{smallmatrix}\right)$ and $\mathbf{A}_{M_2} = \left(\begin{smallmatrix} 5 & -4 \\ 8 & -5 \end{smallmatrix}\right)$. Then both M_1 and M_2 are elliptic and have different fixed points, because $c(A-D)-(a-d)C = 2 \neq 0$ and $b(A-D)-(a-d)B = 6 \neq 0$. The traces satisfy $\operatorname{tr}^2 M_1 = 0$, $\operatorname{tr}^2 M_2 = 3$ and $\operatorname{tr}^2 M_1M_2 = \frac{7}{3}$, hence M_1M_2 is elliptic *with infinite order.*

  This means that if we modify (d1) to $\operatorname{tr}^2 M_1M_2 \in [0,4)$, i.e. M_1M_2 is elliptic but with a general value of trace, we find a solution.

- Let $\mathbf{A}_{M_1} = \left(\begin{smallmatrix} 1 & 10 \\ -5 & -1 \end{smallmatrix}\right)$ and $\mathbf{A}_{M_2} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then both M_1 and M_2 are elliptic and have different fixed points, because $c(A-D)-(a-d)C = 3$ and $b(A-D)-(a-d)B = -8$. The traces satisfy $\operatorname{tr}^2 M_1 = 0$, $\operatorname{tr}^2 M_2 = 1$ and $\operatorname{tr}^2 M_1M_2 = 4$, hence M_1M_2 is parabolic and *not elliptic.*

  This means that if we modify (d1) to $\operatorname{tr}^2 M_1M_2 \in \mathbb{N}_0$, i.e. M_1M_2 is generally not elliptic but has an integer trace, we find a solution.

- Let $\mathbf{A}_{M_1} = \left(\begin{smallmatrix} 1 & 10 \\ -5 & -1 \end{smallmatrix}\right)$ and $\mathbf{A}_{M_2} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then both M_1 and M_2 are elliptic and have *the same fixed point*, because $c(A-D)-(a-d)C = b(A-D)-(a-d)B = 0$. The traces satisfy $\operatorname{tr}^2 M_1 = 3$, $\operatorname{tr}^2 M_2 = 1$ and $\operatorname{tr}^2 M_1M_2 = 3$, hence M_1M_2 is elliptic and has the finite order.

  This means that if we omit (d4), (d5), i.e. we allow M_1 and M_2 to have the same fixed point, we find a solution.

These examples are summarized in Table 5.2, with two additional ones.

Table 5.2. The examples of M_1, M_2 such that the elements of their matrices $\mathbf{A}_{M_1}$ and $\mathbf{A}_{M_2}$ solve some of the conditions. We denote $\omega_c := c(A-D)-(a-d)C$ and $\omega_b := b(A-D)-(a-d)B$.

Equation: **Quantity:** **Condition:**	(a,b,c,d) $\in \mathbb{Z}^4$	(A,B,C,D) $\in \mathbb{Z}^4$	(d1) $\operatorname{tr}^2 M_1$ $\in \{0,\dots,3\}$	(d2) $\operatorname{tr}^2 M_2$ $\in \{0,\dots,3\}$	(d3) $\operatorname{tr}^2 M_1M_2$ $\in \{0,\dots,3\}$	(d4) ω_c $\neq 0$	(d5) ω_b $\neq 0$
	$(6,-3,7,-3)$	$(5,-4,8,-5)$	0	3	**7/3**	2	6
	$(1,10,-5,-1)$	$(0,-1,1,1)$	0	1	**4**	3	-8
	$(4,-1,4,2)$	$(0,1,-4,2)$	3	1	3	**0**	**0**
	$(11,-7,13,19)$	$(-4,-7,13,4)$	0	3	1	**0**	**0**
	$(-1,5,-1,3)$	$(2,-5,1,-2)$	2	2	0	**0**	**0**

5.2. Program for finding solutions of the Diophantine system

In Claim 5.5, we state that the system of Diophantine conditions (d1)–(d5) has no integer solutions with all unknowns in modulus less than or equal to 100.

To show this statement, we used a computer program written in the programming language C++. The program is not long, which allows us to explain its principles in detail.

Headers for text output handling.

```
#include<iostream>
using namespace std;
```

LL is a type we use to store integers. The maximum value of number in LL is (at our machine) over $9 \cdot 10^{18}$, as we checked by LONG_MAX in the module <climits>.

```
typedef long int LL;
```

We use the Euclid algorithm to compute the greatest common divisor of two numbers. We have to deal with the input $\mathtt{a} = \mathtt{b} = 0$, for which the value of gcd is undefined in mathematics; we set the result to 0 in this case, since in the end, we compute $\gcd(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d})$ and if all of them are 0, the matrix is singular and we exclude it no matter the value of gcd.

```
LL gcd(LL a, LL b){
  if(a<0) a=-a;
  if(b<0) b=-b;
  if((a==0)&&(b==0)) return 0;
  if(a==0) return b;
  if(b==0) return a;
  LL c=a%b;
  while(c!=0){
    a=b;
    b=c;
    c=a%b;
  }
  return b;
}
```

The largest numbers we manipulate are squared traces of products of matrices, i.e. numbers of the form $(\mathtt{XX} + \mathtt{XX} + \mathtt{XX} + \mathtt{XX})^2$ where X stands for a coefficient in the matrix. This is in modulus smaller than $16 \cdot \mathtt{M}^4$, where M is the upper bound for the coefficients. We restrict ourselves to $\mathtt{M} \leq 2 \cdot 10^4$, which gives $16 \cdot \mathtt{M}^4 < 3 \cdot 10^{18} <$ `LONG_MAX`.

The maximal choice $\mathtt{M} = 20\,000$ would make our simulation run for ages. Thus we bound the coefficients by much smaller $\mathtt{M} = 100$.

```
const LL M=100;
```

The main program.

```
int main(){
```

We need several integer variables: for the matrices' coefficients, determinants, to store some greatest common divisor (see later) and traces of the matrices. We work with matrices $M_1 = \left(\begin{smallmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{smallmatrix}\right)$.

```
  LL a,b,c,d,A,B,C,D;
  LL xdet,xDET;
  LL xgcd;
  LL t,T,Tt;
```

The main part of the program. It comprises 8 nested for-loops, each for one coefficient of the matrices M_1, M_2. At various places we exclude such combinations of coefficients that cannot lead to a new solution of the equations.

```
  for(a=-M; a<=M; a++){
```

Printing verbose information to enable checking the progress.

```
    cout << "*** a = " << a << " ***" << endl;
    for(b=-M; b<=M; b++){
      xgcd=gcd(a,b);
```

Lemma 5.6, relation $v^{(1)} \in \mathit{Sol} \Leftrightarrow v^{(3)} \in \mathit{Sol}$, gives that if there is a solution with $\mathtt{b} \leq \mathtt{c}$, then there is a solution with $\mathtt{c} \leq \mathtt{b}$ as well. Therefore we can restrict the simulation to $\mathtt{c} \leq \mathtt{b}$.

```
      for(c=-M; c<=b; c++){
        xgcd=gcd(xgcd,c);
```

Lemma 5.6, relation $v^{(1)} \in \mathit{Sol} \Leftrightarrow v^{(4)} \in \mathit{Sol}$, gives that if there is a solution with $\mathtt{a} \leq \mathtt{d}$, then there is a solution with $\mathtt{d} \leq \mathtt{a}$ as well.

```
        for(d=-M; d<=a; d++){
```

In the four outer for-loops, we compute the number $\mathtt{xgcd} = \gcd(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d})$. The 4-tuplets such that $\mathtt{xgcd} > 1$ cannot bring a new solution since for such solution, there must be one for a 4-tuplet $(\mathtt{a}/\mathtt{xgcd}, \mathtt{b}/\mathtt{xgcd}, \mathtt{c}/\mathtt{xgcd}, \mathtt{d}/\mathtt{xgcd})$ as well.

```
          if(gcd(xgcd,d)!=1) continue;
```

The matrix M_1 cannot have negative or zero determinant.

```
          xdet=a*d-b*c;
          if(xdet<=0) continue;
```

Trace squared of the matrix M_1.

```
      t=(a+d)*(a+d);
```

The trace square of the transformation is $\frac{\operatorname{tr}^2 M_1}{\det M_1}$. We exclude such matrices that $\frac{\operatorname{tr}^2 M_1}{\det M_1} \geq 4$, which can be rewritten as $\operatorname{tr}^2 M_1 \geq 4 \det M_1$ to avoid divisions.

```
      if(t>=4*xdet) continue;
```

The trace has to be an integer, which means that the division $\det M_1 \div \operatorname{tr}^2 M_1$ cannot give a non-zero residue.

```
      if((t%xdet)!=0) continue;
```

Lemma 5.6, relation $v^{(1)} \in \mathit{Sol} \Leftrightarrow v^{(5)}, v^{(6)} \in \mathit{Sol}$, allows us to omit negative values of `A` and `B`, since the existence of a solution with negative `A`, `B` enforces the existence of a solution with non-negative values.

```
      for(A=0; A<=M; A++){
        for(B=0; B<=M; B++){
          for(C=-M; C<=M; C++){
```

Lemma 5.6, relation $v^{(1)} \in \mathit{Sol} \Leftrightarrow v^{(2)} \in \mathit{Sol}$, gives that if there is a solution with `d` $\leq$ `D`, then there is a solution with `D` $\leq$ `d` as well.

```
            for(D=-M; D<=d; D++){
```

Now follows conditions on $\det M_2$ and $\operatorname{tr}^2 M_2$, analogous to those for M_1.

```
              xDET=A*D-B*C;
              if(xDET<=0) continue;
              T=(A+D)*(A+D);
              if(T>=4*xDET) continue;
              if((T%xDET)!=0) continue;
```

We compute the trace squared of a matrix $M_1 M_2$ and perform similar restrictions as on M_1 and M_2.

```
              Tt=a*A+b*C+c*B+d*D;
              Tt=Tt*Tt; // trace squared of M1*M2
              if(Tt>=4*xdet*xDET) continue;
              if((Tt%(xdet*xDET))!=0) continue;
```

Finally, we forbid M_1 and M_2 to have the same fixed point.

```
              if(
                (c*(A-D)-(a-d)*C==0)
                &&
                (b*(A-D)-(a-d)*B==0)
                ) continue;
```

When we get to this point, we get a solution of the system which we print.

```
              cout << "solution: a,b,c,d,A,B,C,D = "
                << a << "," << b << ","
                << c << "," << d << ","
```

```
                    << A << "," << B << ","
                    << C << "," << D << ","
                    << endl;
```

Terminate the for-loops and exit the program.

```
  }}}}}}}}
  return 0;
}
```

CHAPTER 6

Conclusions

This thesis studies some aspects of Fuchsian groups and aims on the groups of rational transformations. We have not been able to provide a definite answer to the question of existence of such groups. In Chapter 5, we conjecture that no such groups exist, and we support this conjecture by a computer experiment that is described in Section 5.2. Very important condition is given in Theorem 5.3.

Beside the discussion on rational groups, we proved several interesting results:

- Theorem 4.9, which generalizes the result in [For25];
- Theorem 4.12, which provides a large family of Fuchsian groups that contain no elliptic transformations and their fundamental domain is unbounded;
- Theorem 4.18, which allows to compute the angle of rotation of elliptic transformations fixing the vertices of (generalized) Ford fundamental domains and pre-Ford domains.

6.1. Open problems

The problem of existence of a rational Fuchsian group with bounded fundamental domain remains open. As well, more investigations in the theory of Möbius number systems can be done. Especially, only orientation-preserving Möbius transformations have been considered in these systems. If we add the *orientation-reversing Möbius transformations*, i.e. such $M_{a,b,c,d}$ that $ad-bc<0$, we get the *general Möbius group* $\mathfrak{GM}(2,\mathbb{R})$. The group $\mathfrak{M}(2,\mathbb{R})$ is a subgroup of $\mathfrak{GM}(2,\mathbb{R})$ of the index 2. Expanding to general the Möbius group would allow to use the knowledge on Möbius number systems to the simple continued fraction, and to the radix representations with the negative base, such as the Ito-Sadahiro numeration [IS09].

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Index of Notations

Symbol	Description
$\mathbb{N}$	natural numbers, $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$
$\mathbb{Z}$	integer numbers, $\mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\arg z$	argument of a complex number z, $z = \lvert z\rvert\, e^{i \arg z}$
$\Re z, \Im z$	real and imaginary component of a complex number, $z = \Re z + i\Im z$
$\bar{z}$	complex conjugate of $z \in \mathbb{C}$, $\bar{z} = \Re z - i\Im z$
$\#X$	number of elements of the set X
$\overline{\mathbb{R}}$	extended real line, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$
$\overline{\mathbb{C}}$	extended complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$
$\mathbb{U}$	upper complex half-plane, $\mathbb{U} := \{z \in \mathbb{C} \mid \Im z > 0\}$
$\partial\mathbb{U}$	boundary of $\mathbb{U}$, synonym for $\overline{\mathbb{R}}$
$\mathbb{D}$	unit complex disc, $\mathbb{D} := \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$
$\partial\mathbb{D}$	boundary of $\mathbb{D}$, $\partial\mathbb{D} := \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
$[z_1, z_2]$	geodesic between points z_1 and z_2
$\rho(z_1, z_2)$	hyperbolic distance of two points, the hyperbolic length of $[z_1, z_2]$.
$\widetilde{Q}$	topological closure of Q in $\mathbb{U}$
$\overline{Q}$	topological closure of Q in $\mathbb{U} \cup \partial\mathbb{U}$
$\mathrm{GL}^+(2, \mathbb{R})$	group of 2×2 real matrices with strictly positive determinant
$\mathrm{PGL}^+(2, \mathbb{R})$	group $\mathrm{GL}^+(2, \mathbb{R})/\sim$, where $\mathbf{A} \sim \mathbf{B}$ iff $(\exists\lambda \in \mathbb{R}, \lambda \neq 0)(\mathbf{B} = \lambda\mathbf{A})$
$\mathrm{PSL}(2, \mathbb{Z})$	group of 2×2 integer matrices with determinant 1, factorized by $\{\mathbf{I}, -\mathbf{I}\}$
$\mathfrak{M}(2, \mathbb{R})$	group of all orientation-preserving real Möbius transformations
$\mathfrak{M}(2, \mathbb{Z})$	group of all orientation-preserving real Möbius transformations with integer coefficients
$\mathfrak{GM}(2, \mathbb{R})$	group of all real Möbius transformations
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	typical notation of coefficients of a matrix of a Möbius transformation in the half-plane model
$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$	typical notation of coefficients of a matrix of a Möbius transformation in the disc model
$M^\bullet$	circle derivative of M
$I(M)$	isometric circle of M
$\boldsymbol{V}(M)$	expansion area of M
$\mathbb{X}_F$	convergence space for the system F

Symbol	Description
$\langle g_1, \ldots, g_m \rangle$	group generated by the elements $g_1, \ldots, g_m$
$\mathrm{len}_{g_1, g_2, \ldots, g_k}(h)$	length of $h \in \langle g_1, \ldots, g_m \rangle$
$\mathcal{A}$	alphabet, in general any finite set with $\#\mathcal{A} \geq 2$
$\mathcal{A}^*$	the set of finite words over the alphabet $\mathcal{A}$
u^*	the set of words of the form u^k for $k \in \mathbb{N}$
$\mathcal{A}^{\mathbb{N}}$	the set of right infinite words over the alphabet $\mathcal{A}$
$u^{\mathbb{N}}$	the ultimately periodic infinite word $u^{\mathbb{N}} = uuu\cdots$
Σ_X	subshift of finite type with forbidden set of factors X

General Index

Colophon

This thesis was typeset in the $\LaTeX\,2_\varepsilon$ typesetting system, with `report` class and additional packages `amsfonts`, `amsmath`, `amssymb`, `amsthm`, `babel`, `bm`, `booktabs`, `fancyhdr`, `floatrow`, `fontenc`, `geometry`, `graphicx`, `inputenc`, `ltablex`, `makeidx`, `mathtools`, `nicefrac`, `relsize`, `rotating`, `tikz`, `titlesec`, `tocloft`, `url`, `xcolor`, as well as with various author's customizations. The figures were drawn by hand or using the Maple software. The typefaces used are Computer Modern and Antykwa Toruńska. The source files of the thesis as well as all the figures can be found on the website `http://kmlinux.fjfi.cvut.cz/~hejdato1`

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