ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

Fakulta jaderná a fyzikálně inženýrská

Katedra matematiky
Obor: Matematické inženýrství
Zaměření: Matematické modelování

Morphisms preserving the set of words coding three interval exchange

Morfismy zachovávající slova kódující výměnu tří intervalů

VÝZKUMNÝ ÚKOL

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Rok: **2010**

Poděkování

Chtěl bych poděkovat paní profesorce Editě Pelantové a dalším členům skupiny TIGR na katedře matematiky za kvalitní vedení mé práce na tomto výzkumném úkolu, a také za velmi příjemné a plodné pracovní prostředí.

Tomáš Hejda

Prohlášení

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V Praze dne 21. června 2010 Tomáš Hejda

Title: Morphisms preserving the set of words coding three interval exchange

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Keywords: interval exchange, amicable Sturmian morphisms, incidence matrix of morphism

Abstract

Any amicable pair φ, ψ of Sturmian morphisms enables a construction of a ternary morphism η which preserves the set of infinite words coding 3-interval exchange. We determine the number of amicable pairs with the same incidence matrix in $SL(2, \mathbb{N})$ and we study incidence matrices associated with the corresponding ternary morphisms η.

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1 Introduction

Sturmian words are well-described objects in combinatorics on words. They can be defined in several equivalent ways [5], e.g. as words coding a two-interval exchange transformation with irrational ratio of lengths of the intervals. Morphisms preserving the set of Sturmian words are called *Sturmian* and they form a monoid generated by three of its elements (see [6, 12]). Let us denote this monoid by $\mathcal{M}_{\text{Sturm}}$.

In this paper, we consider morphisms preserving the set of words coding a three-interval exchange transformation with permutation $(3, 2, 1)$, the so-called *3iet words*. We call these morphisms *3iet-preserving*. Monoid of these morphisms, denoted by $\mathcal{M}_{\text{3iet}}$, is not fully described. It is shown (see [9]) that the monoid $\mathcal{M}_{\text{3iet}}$ is not finitely generated. Recently, in [2], pairs of amicable Sturmian morphisms were defined. The authors used this notion to describe morphisms that have as a fixed point a non-degenerate 3iet word, i.e. word with complexity $\mathcal{C}(n) = 2n + 1$. Using the operation of "ternarization", we can assign a morphism $\eta = \text{ter}(\varphi, \psi)$ over a ternary alphabet to a pair of amicable Sturmian morphisms. We show that such η is a 3iet-preserving morphism. Moreover, we show that the set

$$\mathcal{M}_{\text{ter}} = \big\{\text{ter}(\varphi, \psi)\big|\varphi, \psi \text{ amicable morphisms}\big\} \tag{1}$$

is a monoid, but it does not cover the whole monoid $\mathcal{M}_{\text{3iet}}$.

We also study the incidence matrices of morphisms $\eta \in \mathcal{M}_{\text{ter}}$. From the definition of amicable Sturmian morphisms φ, ψ we can derive that φ and ψ have the same incidence matrix $\mathbf{A} \in \mathbb{N}^{2\times 2}$, where $\det \mathbf{A} = \pm 1$. As shown in [13], for every matrix $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ with $\det \mathbf{A} = \pm 1$, there exist $p_0 + p_1 + q_0 + p_1 - 1$ Sturmian morphisms. We will show the following theorem concerning the number of pairs of amicable Sturmian morphisms with a given matrix.

Theorem 1. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$. *Then there exist exactly*

$$m\big(\|\mathbf{A}\| - 1\big) + \frac{m}{2}\big(\det \mathbf{A} - m\big) \tag{2}$$

pairs of amicable Sturmian morphisms with incidence matrix $\mathbf{A}$, *where* $m = \min\{p_0 + p_1, q_0 + q_1\}$ *and* $\|\mathbf{A}\| = p_0 + p_1 + q_0 + q_1$.

Moreover, for a given matrix $\mathbf{A}$, we will describe all matrices $\mathbf{B} \in \mathbb{N}^{3\times 3}$ such that $\mathbf{B}$ is an incidence matrix of $\eta = \text{ter}(\varphi, \psi)$ for amicable Sturmian morphisms φ, ψ with incidence matrix $\mathbf{A}$.

2 Preliminaries

2.1 Words over a finite alphabet

Besides the infinite words, we consider *finite words* over the alphabet $\mathcal{A}$. We write $w = w_0 w_1 \cdots w_{n-1}$, where $w_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, $i < n$. We denote by $|w|$ the length n of the finite word w. We denote by $|w|_a$ the number of occurrences of a letter $a \in \mathcal{A}$ in the word w. The set of all finite words on the alphabet $\mathcal{A}$ including the empty word ϵ is denoted by $\mathcal{A}^*$. The set $\mathcal{A}^*$ with the operation of concatenation is a monoid. On the set $\mathcal{A}^*$ we define a relation of *conjugation*: $w \sim w'$, if there exists $v \in \mathcal{A}^*$ such that $wv = vw'$. A *morphism* from $\mathcal{A}^*$ to $\mathcal{B}^*$ is a mapping $\varphi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. It is clear that a morphism is well defined by images of letters $\varphi(a)$ for all $a \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}$, then φ is called a *morphism over* $\mathcal{A}$.

The set of *infinite words* over the alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$. The action of a morphism can be naturally extended to an infinite word $(u_i)_{i\in\mathbb{N}}$ putting $\varphi(u) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$. If an infinite word $u \in \mathcal{A}^{\mathbb{N}}$ satisfies $\varphi(u) = u$, we call it a *fixed point* of the morphism φ over $\mathcal{A}$.

To a morphism φ over $\mathcal{A}$ we assign an *incidence matrix* $\mathbf{M}_\varphi$ defined by $(\mathbf{M}_\varphi)_{ab} = |\varphi(a)|_b$ for all $a, b \in \mathcal{A}$. To a finite word $v \in \mathcal{A}^*$ we assign a *Parikh vector* $\mathbf{M}_v$ defined by $(\mathbf{M}_v)_b = |v|_b$ for all $b \in \mathcal{A}$.

The *language* of an infinite word u is the set of all its factors. Let us recall that a finite word $w \in \mathcal{A}^*$ is a *factor* of $u = (u_i)_{i\in\mathbb{N}}$, if there exist indices $n, j \in \mathbb{N}$ such that $w = u_n u_{n+1} \cdots u_{n+j-1}$. The language of an infinite word is denoted by $\mathcal{L}(u)$.

It is known that the language of neither Sturmian nor 3iet word depends on the point $x_0 \in [0, 1)$, the orbit of which the infinite word codes. It depends only on slope ε or parameters α, β.

The *(factor) complexity* of an infinite word u is a mapping $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$, which returns the number of factors of u of the length n, thus $\mathcal{C}_u(n) = \#\{w \in \mathcal{L}(u) \big| |w| = n\}$. It is easy to see that a word u is periodic if and only if there exists $n_0 \in \mathbb{N}$ such that $C_u(n_0) \leq n_0$.

2.2 Interval exchange

We consider Sturmian words, i.e. aperiodic words given by exchange of 2 intervals with permutation $(2, 1)$, and words given by exchange of 3 intervals with permutation $(3, 2, 1)$. Let us recall that general r-interval exchange transformations were introduced already in [11].

Two-interval exchange. The 2-interval exchange transformation S is a mapping $S : [0, 1) \to [0, 1)$. It is determined by its slope $\varepsilon \in [0, 1]$ and is given by

$$Sx = \begin{cases} x + 1 - \varepsilon & \text{if } x \in [0, \varepsilon) \\ x - \varepsilon & \text{if } x \in [\varepsilon, 1). \end{cases}$$

The orbit of a point $x_0 \in [0,1)$ with respect to the transformation S, i.e. the sequence $x_0, Sx_0, S^2x_0, \ldots$ can be coded by an infinite word $u = (u_i)_{i=0}^{\infty}$ on the binary alphabet $\{0,1\}$. The infinite word is given by

$$u_i = \begin{cases} 0 & \text{if } S^i x_0 \in [0, \varepsilon), \\ 1 & \text{if } S^i x_0 \in [\varepsilon, 1). \end{cases} \tag{3}$$

It is a well-known fact that for an irrational ε, the word u is Sturmian. Using the same construction on the partition of the interval $(0,1]$ into $(0,\varepsilon] \cup (\varepsilon, 1]$, we again obtain a Sturmian word. On the other hand, every Sturmian word can be obtained by one of the above two constructions. The set of Sturmian words will be denoted by $\mathcal{W}_{\text{Sturm}}$.

In [12], the authors show that Sturmian words are the aperiodic words with minimal complexity, i.e. $\mathcal{C}_u(n) = n + 1$ for all $u \in \mathcal{W}_{\text{Sturm}}$ and $n \in \mathbb{N}$. We can see that

$$S^i x_0 = \{x_0 - i\varepsilon\} \quad \text{for all} \quad x_0 \in [0,1), \tag{4}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the *fractional part* of a number $x \in \mathbb{R}$. Then $u_i = \lfloor x_0 - i\varepsilon \rfloor - \lfloor x_0 - (i+1)\varepsilon \rfloor$, which is exactly the formula how [12] define mechanical words.

We will use another fact about two-interval exchange. Let $\varphi \in \mathcal{M}_{\text{Sturm}}$ be a Sturmian morphism. Then the word $\varphi(a)$ for $a \in \{0,1\}$ codes two-interval exchange with the slope $\frac{|v|_0}{|v|}$. We should see this from the Lemma 2.1.15 in [12]. The word a^k is a factor of some Sturmian word, hence the word $\varphi(a)^k$ is balanced for any $k \in \mathbb{N}$, which means that the infinite word $u = \varphi(a)^\omega = \varphi(a)\varphi(a)\varphi(a)\cdots$ is balanced and periodic, thus it is rational mechanical. In our terms, this means that it codes a rational 2-interval exchange; it is as well shown there that the slope of the transformation is exactly $\frac{|v|_0}{|v|}$.

Three-interval exchange. The 3-interval exchange transformation T is determined by two parameters $\alpha, \beta \in (0,1)$ satisfying $\alpha + \beta < 1$. Using parameters α, β and $\gamma = 1 - \alpha - \beta$ we partition the interval $[0,1)$ into $I_A = [0, \alpha)$, $I_B = [\alpha, \alpha + \beta)$ and $I_C = [\alpha + \beta, 1)$. The mapping T is given by

$$Tx = \begin{cases} x + \beta + \gamma & \text{if } x \in I_A, \\ x - \alpha + \gamma & \text{if } x \in I_B, \\ x - \alpha - \beta & \text{if } x \in I_C. \end{cases}$$

The orbit of a point $x_0 \in [0,1)$ with respect to the transformation T is coded by a word $u = (u_i)_{i=0}^{\infty}$ over the ternary alphabet $\{A, B, C\}$:

$$u_i = X \quad \text{if} \quad T^i x_0 \in I_X.$$

Similarly to the case of 2-interval exchange transformation, we can define the exchange of 3 intervals using the partition $(0,1] = (0,\alpha] \cup (\alpha, \alpha+\beta] \cup (\alpha+\beta, 1]$. If $\frac{1-\alpha}{1+\beta}$ is irrational, the infinite word u is aperiodic, and we call it a *3iet word*; the set of these words is denoted by $\mathcal{W}_{3\text{iet}}$. For combinatorial properties of 3iet words, see [8].

Aperiodic words coding 3-interval exchange transformations, called here 3iet words, have the complexity $\mathcal{C}_u(n) \leq 2n+1$ for all $n \in \mathbb{N}$. If a 3iet word $u \in \mathcal{W}_{3\text{iet}}$ satisfies $\mathcal{C}_u(n) = 2n+1$ for all $n \in \mathbb{N}$, we call it a *non-degenerate* 3iet word; otherwise we call it a *degenerate* 3iet word and it is a quasi-Sturmian word (see [7]).

2.3 Standard pairs and standard morphisms

In [13], the notion of standard pairs is introduced. If we define two operators on pairs of words $L, R : \mathcal{A}^* \times \mathcal{A}^* \to \mathcal{A}^* \times \mathcal{A}^*$ as

$$L(x,y) = (x, xy), \qquad R(x,y) = (yx, y),$$

we say that a pair (x,y) is a *standard pair*, if it can be obtained from the pair $(0,1)$ by applying the operators L and R finitely many times.

We say that a binary morphism φ is *standard*, if there exists a standard pair (x,y) such that

$$\begin{aligned} \varphi(0) &= x, \\ \varphi(1) &= y, \end{aligned} \qquad \text{or} \qquad \begin{aligned} \varphi(0) &= y, \\ \varphi(1) &= x. \end{aligned}$$

The authors of [13] show the close connection between the standard morphisms and all the Sturmian morphisms:

1. Every standard morphism is Sturmian.

2. For every matrix $\mathbf{A} \in \mathbb{N}^{2\times 2}$ with $\det \mathbf{A} = \pm 1$, there exists exactly one standard morphism φ with incidence matrix $\mathbf{M}_\varphi = \mathbf{A}$.

3. Every Sturmian morphism $\psi \in \mathcal{M}_{\text{Sturm}}$ is a right conjugate to some standard morphism φ. Let us recall that a morphism ψ over $\mathcal{A}$ is a *right conjugate* to φ, if there exists a finite word $v \in \mathcal{A}^*$ such that

$$\psi(a)v = v\varphi(a) \quad \text{for all letters} \quad a \in \mathcal{A}.$$

2.4 Amicable words and morphisms

In the article [4], authors show the close connection between 3iet and Sturmian words using two morphisms $\sigma_{01}, \sigma_{10} : \{A, B, C\}^* \to \{0, 1\}^*$ given by

$$\begin{aligned} \sigma_{01}(A) &= 0, & \sigma_{10}(A) &= 0, \\ \sigma_{01}(B) &= 01, & \sigma_{10}(B) &= 10, \\ \sigma_{01}(C) &= 1, & \sigma_{10}(C) &= 1. \end{aligned}$$

In [4], the following theorem is proved.

Theorem 2. *An infinite ternary word $u \in \{A, B, C\}^{\mathbb{N}}$ is a 3iet word if and only if the words $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian.*

This theorem motivated the authors of [1] to introduce the relation of amicability of words.

Definition 3. Let $w, w' \in \{0, 1\}^*$, let $b \in \mathbb{N}$. We say that w is *b-amicable* to w', if there exists a factor $v \in \{A, B, C\}^*$ of some 3iet word such that

$$w = \sigma_{01}(v), \qquad w' = \sigma_{10}(v) \quad \text{and} \quad |v|_B = b.$$

We say that w is *amicable* to w', if w is b-amicable to w' for some $b \in \mathbb{N}$, and we denote it by $w \propto w'$.

The ternary word v is called a *ternarization* of w and w', and we write $v = \mathrm{ter}(w, w')$.

It is easy to see that if $w \propto w'$, then they are factors of the same Sturmian word and their Parikh vectors coincide.

In [1], the notion of "amicable words" plays a crucial role in enumeration of words with length n occurring in a 3iet word. In [2], the authors investigate ternary morphisms that have a non-degenerate 3iet fixed point using the following notion of amicability of two Sturmian morphisms.

Definition 4. Let φ, ψ be Sturmian morphisms over the alphabet $\{0, 1\}$. We say that φ is *amicable* to ψ, if

$$\begin{aligned} \varphi(0) &\propto \psi(0), \\ \varphi(01) &\propto \psi(10) \\ \text{and} \quad \varphi(1) &\propto \psi(1). \end{aligned}$$

We denote this relation by $\varphi \propto \psi$. The morphism η over the ternary alphabet $\{A, B, C\}$, given by

$$\begin{aligned} \eta(A) &= \mathrm{ter}(\varphi(0), \psi(0)), \\ \eta(B) &= \mathrm{ter}(\varphi(01), \psi(10)), \\ \eta(C) &= \mathrm{ter}(\varphi(1), \psi(1)) \end{aligned}$$

is called the *ternarization* of morphisms φ and ψ, and is denoted by $\eta = \mathrm{ter}(\varphi, \psi)$. Set of these η is denoted by $\mathcal{M}_{\mathrm{ter}}$.

The article [2] states the following theorem:

Theorem 5. *Let η be a ternary morphism with non-degenerate 3iet fixed point. Then $\eta \in \mathcal{M}_{\mathrm{ter}}$ or $\eta^2 \in \mathcal{M}_{\mathrm{ter}}$.*

3 Main results

3.1 Globally 3iet-preserving morphisms

Analogously to the terminology introduced for Sturmian words and morphisms in [6], the ternarization η, having a 3iet fixed point, is *locally 3iet-preserving*, i.e. there exists $u \in \mathcal{W}_{3\mathrm{iet}}$ such that $\eta(u) \in \mathcal{W}_{3\mathrm{iet}}$. We now prove a partial result about *(globally) 3iet-preserving* morphisms, i.e. ternary morphisms η such that

$$\eta(u) \in \mathcal{W}_{3\mathrm{iet}} \quad \text{for all} \quad u \in \mathcal{W}_{3\mathrm{iet}}.$$

Proposition 6. *Let $\eta = \mathrm{ter}(\varphi, \psi)$ for amicable Sturmian morphisms $\varphi \propto \psi$. Then η is a globally 3iet-preserving morphism.*

Proof. Directly from definitions we see that

$$\begin{aligned} &\sigma_{01}\eta(A) = \varphi(0), && \sigma_{01}\eta(B) = \varphi(01), && \sigma_{01}\eta(C) = \varphi(1), \\ &\sigma_{10}\eta(A) = \psi(0), && \sigma_{10}\eta(B) = \psi(10), && \sigma_{10}\eta(C) = \psi(1). \end{aligned}$$

Therefore

$$\sigma_{01}\eta(v) = \varphi\sigma_{01}(v) \qquad \text{and} \qquad \sigma_{10}\eta(v) = \psi\sigma_{10}(v) \tag{5}$$

for any factor v of a 3iet word $u \in \mathcal{W}_{3\mathrm{iet}}$. According to Theorem 2 we get that $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian words, and since φ and ψ are Sturmian morphisms, we obtain that $\sigma_{01}\eta(u)$ and $\sigma_{10}\eta(u)$ are Sturmian words as well, which means, according to the same theorem, that the word $\eta(u)$ is 3iet. □

Proposition 7. *Let $\varphi_i \propto \psi_i$ be Sturmian morphisms, for $i = 1, 2$. Then*

$$\mathrm{ter}(\varphi_1, \psi_1) \circ \mathrm{ter}(\varphi_2, \psi_2) = \mathrm{ter}(\varphi_1 \circ \varphi_2, \psi_1 \circ \psi_2).$$

Proof. It can be shown that the relation of amicability is preserved by composition of morphisms. More precisely $\varphi_1\varphi_2 \propto \psi_1\psi_2$. Denote $\eta_1 = \mathrm{ter}(\varphi_1, \psi_1)$, $\eta_2 = \mathrm{ter}(\varphi_2, \psi_2)$. Using the relation (5), we see that for all $v \in \{A, B, C\}^*$

$$\begin{aligned} \sigma_{01}\eta_1\eta_2(v) &= \varphi_1\sigma_{01}\eta_2(v) = \varphi_1\varphi_2\sigma_{01}(v) \\ \text{and} \quad \sigma_{10}\eta_1\eta_2(v) &= \psi_1\sigma_{10}\eta_2(v) = \psi_1\psi_2\sigma_{10}(v). \end{aligned}$$

But this means that $\eta_1\eta_2 = \mathrm{ter}(\varphi_1\varphi_2, \psi_1\psi_2)$. □

As a consequence of previous two propositions, we can state the following theorem.

Theorem 8. *The set $\mathcal{M}_{\rm ter}$ of all ternarizations of amicable Sturmian morphisms with the operation of composition of morphisms is a sub-monoid of the monoid $\mathcal{M}_{\rm 3iet}$ of all globally 3iet-preserving morphisms.*

Unfortunately, $\mathcal{M}_{\rm ter} \subsetneq \mathcal{M}_{\rm 3iet}$. Consider for example the morphism

$$\eta(A) = B, \qquad \eta(B) = CAC, \qquad \eta(C) = C. \tag{6}$$

As shown in [9], this morphism is 3iet-preserving, but it can be easily verified that it is not a ternarization of any pair of Sturmian morphisms, using the following statement.

Proposition 9. *A ternary morphism η is a ternarization, i.e. $\eta \in \mathcal{M}_{\rm ter}$, if and only if it satisfies*

$$\sigma_{01}\eta(B) = \sigma_{01}\eta(AC) \quad \text{and} \quad \sigma_{10}\eta(B) = \sigma_{10}\eta(CA).$$

Proof. The implication ($\Rightarrow$). Suppose $\eta = \mathrm{ter}(\varphi, \psi)$. According to (5) we get

$$\sigma_{01}\eta(B) = \varphi\sigma_{01}(B) = \varphi(01) = \varphi\sigma_{01}(AC) = \sigma_{01}\eta(AC),$$
$$\sigma_{10}\eta(B) = \psi\sigma_{10}(B) = \psi(10) = \psi\sigma_{10}(CA) = \sigma_{10}\eta(CA).$$

The implication ($\Leftarrow$). Define morphisms φ, ψ as

$$\varphi(0) = \sigma_{01}\eta(A), \qquad \psi(0) = \sigma_{10}\eta(A),$$
$$\varphi(1) = \sigma_{01}\eta(C), \qquad \psi(1) = \sigma_{10}\eta(C).$$

Immediately we get $\mathrm{ter}(\varphi(0), \psi(0)) = \eta(A)$ and $\mathrm{ter}(\varphi(1), \psi(1)) = \eta(C)$. The words $\varphi(01)$ and $\psi(10)$ satisfy $\varphi(01) = \sigma_{01}\eta(AC) = \sigma_{01}\eta(B)$ and $\psi(10) = \sigma_{10}\eta(CA) = \sigma_{10}\eta(B)$, which means that $\mathrm{ter}(\varphi(01), \psi(10)) = \eta(B)$. □

For the morphism (6), we get $\sigma_{01}\eta(B) = 010 \neq 001 = \sigma_{01}\eta(AC)$. Another even simpler example of a 3iet-preserving morphism that is not a ternarization is the morphism interchanging the letters A and C.

3.2 Pairs of amicable Sturmian morphisms

Now, our goal will be to determine the number of amicable pairs of morphisms with incidence matrix $\mathbf{A}$ of $\det \mathbf{A} = \pm 1$. We will use the notion of b-amicable morphisms.

Definition 10. Let φ and ψ be binary morphisms and let $b \in \mathbb{N}$. We say that φ is *b-amicable* to ψ, if φ is amicable to ψ and the number of occurrences of B in $\mathrm{ter}(\varphi(01), \psi(10))$ is b.

We now determine the numbers of pairs of b-amicable Sturmian morphisms. The following proposition and the Theorem 1 were already proven in [10]. We provide a more straightforward proof.

Proposition 11. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$ *and* $b \in \mathbb{N}$. *Put* $p = p_0 + p_1$, $q = q_0 + q_1$. *Then the number* $c_{\mathbf{A}}(b)$ *of pairs of* b*-amicable morphisms with matrix* $\mathbf{A}$ *is equal to*

$$c_{\mathbf{A}}(b) = \begin{cases} \|\mathbf{A}\| - b & \text{if } \det \mathbf{A} = +1 \text{ and } 1 \le b \le \min\{p,q\}, \\ \|\mathbf{A}\| - b - 2 & \text{if } \det \mathbf{A} = -1 \text{ and } 0 \le b \le \min\{p,q\} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

First, let us state the following lemma.

Lemma 1. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$ *and* $b \in \mathbb{N}$. *Put* $p = p_0 + p_1$, $q = q_0 + q_1$ *and* $N = p + q$. *Let* S *be a two-interval exchange with the slope* $p/(p+q)$. *Let* $w^{(k)}$ *be a word of the length* N *that codes* S *with the start point* k/N, *for* $k \in \{0, \dots, N-1\}$.

Then $w^{(k)}$ *is* b*-amicable to* $w^{(\bar{k})}$ *if and only if* $0 \le b \le \min\{p,q\}$ *and* $\bar{k} - k = b$.

Proof. Using (4), we see that $S^i(k/N) \equiv (k - ip)/N \pmod 1$, which is equivalent to $NS^i(k/N) \equiv k - ip \pmod N$. We know that the numbers p and N are co-prime, thus the mapping $f_k : \{0, \dots, N-1\} \to \{0, \dots, N-1\}$ given by the congruence $f_k(i) \equiv k - ip \pmod N$ is a bijection. As well, $f_{\bar{k}}(i) - f_k(i) \equiv \bar{k} - k \pmod N$.

Denote $m = \min\{p,q\}$ and $b = \bar{k} - k$. Consider the following cases:

($b < 0$) We shall see that $w^{(k)}$ is lexicographically larger than $w^{(\bar{k})}$, i.e. if $i \in \mathbb{N}$ is the first position such that $w_i^{(k)} \neq w_i^{(\bar{k})}$, then $w_i^{(k)} = 1$ and $w_i^{(\bar{k})} = 0$. Directly from the definition of amicability, if $w^{(k)} \propto w^{(\bar{k})}$ and $w^{(k)} \neq w^{(\bar{k})}$, then $w^{(k)}$ is lexicographically smaller than $w^{(\bar{k})}$. These two facts make a contradiction.

($b \in \{0, \dots, m\}$) Let $I_a \subset \{0, \dots, N-1\}$ be a set of indices i such that $w_i^{(k)} = a$ and $w_i^{(\bar{k})} \neq a$, for both $a = 0, 1$. To show that $w^{(k)}$ is b-amicable to $w^{(\bar{k})}$, we need to show that $i \in I_0$ implies $i + 1 \in I_1$ and $\# I_0 = \# I_1 = b$. The fact that $\left|w^{(k)}\right|_0 = \left|w^{(\bar{k})}\right|_0$ follows to $\# I_0 = \# I_1$.

Let i be an index such that $f_k(i) \in [p - b, p)$, thus $w_i^{(k)} = 0$. Then $f_{\bar{k}}(i) \in [p, p + b)$, thus $w_i^{(\bar{k})} = 1$. This means $i \in I_0$. For these i, we have $f_k(i + 1) \in [N - b, N)$ and $f_{\bar{k}}(i + 1) \in [0, b)$, which means $i \in I_1$. There are exactly b such indices i.

It remains to show that we covered the whole set I_0. Suppose $f_k(i) < p - b$, then $f_{\bar{k}}(i) < p$ and $w_i^{(\bar{k})} = 0$, which means $i \notin I_0$. Suppose $f_k(i) \ge p$, then $w_i^{(k)} = 1$, which means $i \notin I_0$.

($b \in \{m+1, \dots, N-m-1\}$) Let i be such index that $f_k(i) = p-1$.

Suppose $p \le q$. Then $f_k(i+1) = N-1$, $f_{\bar{k}}(i) = b+p-1$ and $f_{\bar{k}}(i+1) = b-1$, which means that $w_i^{(k)} w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})} w_{i+1}^{(\bar{k})} = 11$.

Suppose $p > q$. Then $f_k(i+1) = N-1$, $f_{\bar{k}}(i) = b-q-1$ and $f_{\bar{k}}(i+1) = b-1$, which means that $w_i^{(k)} w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})} w_{i+1}^{(\bar{k})} = 00$.

Both these are in contradiction with $w^{(k)} \propto w^{(\bar{k})}$.

($b \in \{N-m, \dots, N-1\}$) We can see that $N - m = p + q - \min\{p,q\} = \max\{p,q\}$.

Suppose $p < q$. We will show that $j = 2p$ solves the inequalities

$$\begin{aligned} &p \le j < N, && p \le j + b - N < N, \\ &p \le j - p < N, && 0 \le j + b - p - N < p. \end{aligned}$$

We have $2p > p$; $2p < p + q = N$; $2p + b - N \ge 2p + q - N = p$; $2p+b-N < 2p < N$; $2p-p = p \ge p$; $2p-p = p < N$; $2p+b-p-N = p-(N-b) < p$; $2p+b-p-N = b-(N-p) = b-q \ge 0$.

Let i be index such that $f_k(i) = j$. Then the previous inequalities give $w_i^{(k)} w_{i+1}^{(k)} = 11$ and $w_i^{(\bar{k})} w_{i+1}^{(\bar{k})} = 10$, which is in contradiction with $w^{(k)} \propto w^{(\bar{k})}$.

Suppose $p > q$. We will show that $j = \max\{2p - N, N - b\}$ solves the inequalities

$$\begin{aligned} &0 \le j < p, && 0 \le j + b - N < p, \\ &p \le j - p + N < N, && 0 \le j + b - p < p. \end{aligned}$$

We have $j \ge N - b > 0$, thus $j > 0$; $2p - N = p + (p - N) < p$ and $N - b < p$, thus $j < p$; $j \ge N - b$, thus $j + b - N \ge 0$; $(2p-N)+b-N = p - q - (N-b) < p$ and $(N-b)+b-N = 0 < p$, thus $j+b-N < p$; $j \ge 2p-N$, thus $j-p+N \ge p$; $j-p < 0$, thus $j-p+N < N$; $j \ge N-b$, thus $j+b-p \ge N-b+b-p = q > 0$; $(2p-N)+b-p < 2p-b+b-p = p$ and $(N-b)+b-p = q < p$, thus $j+b-p < p$.

Let i be index such that $f_k(i) = j$. Then the previous inequalities give $w_i^{(k)} w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})} w_{i+1}^{(\bar{k})} = 00$, which is contradiction with $w^{(k)} \propto w^{(\bar{k})}$. □

Proof of Proposition 11. Let S be a 2-interval exchange transformation with the slope $\varepsilon = p/N$. Let $k \in \mathbb{Z}$ and denote $w^{(k)}$ the word of the length $N = \|\mathbf{A}\|$ that codes the orbit of the point $\{k/N\}$ with respect to S. We know that for every Sturmian morphism φ with $\mathbf{M}_\varphi = \mathbf{A}$, there exists

$k \in \{0, \ldots, N-1\}$ such that $\varphi(01) = w^{(k)}$, we will denote this morphism $\varphi^{(k)}$.

Let φ_{std} be a standard morphism with $\mathbf{M}_{\varphi_{\text{std}}} = \mathbf{A}$. Every Sturmian morphism $\varphi^{(k)}$ is a right conjugate to φ_{std}, which means that there exist words $v, v' \in \{0,1\}*$ such that

$$\varphi(aa') = v01v' \quad \text{and} \quad \varphi(a'a) = v10v',$$

where letters a, a' satisfy $aa' = 01$ for $\det \mathbf{A} = +1$ and $aa' = 10$ for $\det \mathbf{A} = -1$. This gives that $\varphi(aa')$ is 1-amicable to $\varphi(a'a)$.

Morphism $\varphi^{(k)}$ is b-amicable to $\varphi^{(\bar{k})}$ if and only if the following conditions are satisfied:

1. $\varphi^{(k)}(01)$ is b-amicable to $\varphi^{(\bar{k})}(10)$;
2. $\varphi^{(k)}(01)$ is amicable to $\varphi^{(\bar{k})}(01)$;
3. Parikh vectors satisfy $\mathbf{M}_{\varphi^{(k)}(0)} = \mathbf{M}_{\varphi^{(\bar{k})}(0)}$.

The 2nd and 3rd conditions assures that $\varphi^{(k)}(0) \propto \varphi^{(\bar{k})}(0)$ and $\varphi^{(k)}(1) \propto \varphi^{(\bar{k})}(1)$.

Let us discuss the cases $\det \mathbf{A} = +1$ and $\det \mathbf{A} = -1$.

($\det \mathbf{A} = +1$) We know that $\varphi^{(k)}(01)$ is 1-amicable to $\varphi^{(k)}(10)$, which implies $\varphi^{(k)}(10) = w^{(k+1)}$. This excludes $k = N-1$.

The 3rd condition is immediately satisfied by $\mathbf{M}_{\varphi^{(k)}} = \mathbf{M}_{\varphi^{(\bar{k})}}$. To satisfy the 1st condition, we need $(\bar{k}+1) - k = b$. To satisfy the 2nd condition, we need $0 \leq \bar{k} - k \leq \min\{p,q\}$. These facts gives $0 \leq k \leq \bar{k} \leq N-2$ and $1 \leq b \leq \min\{p,q\}$. For each such b, we have exactly $N - b$ pairs of such indices $(k, \bar{k})$.

($\det \mathbf{A} = -1$) We know that $\varphi^{(k)}(10)$ is 1-amicable to $\varphi^{(k)}(01)$, which implies $\varphi^{(k)}(10) = w^{(k-1)}$. This excludes $k = 0$.

The 3rd condition is immediately satisfied by $\mathbf{M}_{\varphi^{(k)}} = \mathbf{M}_{\varphi^{(\bar{k})}}$. To satisfy the 1st condition, we need $(\bar{k}-1) - k = b$. To satisfy the 2nd condition, we need $0 \leq \bar{k} - k \leq \min\{p,q\}$. These facts gives $1 \leq k \leq \bar{k} \leq N-1$ and $0 \leq b \leq \min\{p,q\} - 1$. For each such b, we have exactly $N - b - 2$ pairs of such indices $(k, \bar{k})$. □

Proof of Theorem 1. The formula (2) can be obtained by summation of numbers $c_{\mathbf{A}}(b)$ from the previous proposition. □

3.3 Matrices of ternarizations

To each pair of amicable Sturmian morphisms, an incidence matrix of its ternarization is assigned. We now fully describe which matrices from $\mathbb{N}^{3\times 3}$ are matrices of ternarizations.

Theorem 12. *A matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ is the incidence matrix of the ternarization of a pair of amicable Sturmian morphisms if and only if there exist matrix $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ with $\det \mathbf{A} = \Delta = \pm 1$ and numbers $b_0, b_1 \in \mathbb{N}$ such that*

(a) $\left|\frac{b_0(p_1+q_1)-b_1(p_0+q_0)}{p_0+q_0+p_1+q_1}\right| < 1,$

(b) $\frac{1-\Delta}{2} \leq b_0 + b_1 \leq \min\{p_0+p_1, q_0+q_1\} - \frac{\Delta+1}{2},$

(c) $\mathbf{B} = \mathbf{P}\begin{pmatrix} \mathbf{A} & \begin{smallmatrix} b_0 \\ b_1 \end{smallmatrix} \\ 0\ 0 & \Delta \end{pmatrix}\mathbf{P}^{-1}$, *where* $\mathbf{P} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

Proof of the implication $(\Rightarrow)$. Let us denote $p = p_0 + p_1$, $q = q_0 + q_1$, $N = p + q$ and $b = b_0 + b_1 + \Delta$. Then we can see that condition (c) gives

$$\mathbf{B} = \begin{pmatrix} p_0 - b_0 & b_0 & q_0 - b_0 \\ p - b & b & q - b \\ p_1 - b_1 & b_1 & q_1 - b_1 \end{pmatrix}. \tag{7}$$

The fact that (c) is necessary for $\mathbf{B}$ to be an incidence matrix of a ternarization is shown in [3]. Condition (b) is necessary according to Proposition 11, so we only need to show that (a) is satisfied for the matrix of the ternarization $\eta = \mathrm{ter}(\varphi, \psi)$ of a pair of amicable Sturmian morphisms $\varphi \propto \psi$.

We can see that $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ is necessarily an incidence matrix of both φ and ψ. Let S be a 2-interval exchange transformation with rational slope $\varepsilon = p/N$. Then there exist $k, \bar{k} \in \{0, \ldots, N-2\}$ such that $\varphi(01)$, $\psi(01)$ code transformation S with start points $x_0 = k/N$, $\bar{x}_0 = \bar{k}/N$; moreover, $\bar{k} - k = b - \Delta$. We need to determine the value of $b_0 = |\mathrm{ter}(\varphi(0), \psi(0))|_B$. The number b_0 is equal to the number of indices $i \in \{0, 1, \ldots, p_0 + q_0 - 1\}$ such that $S^i x_0 \in \big[(p - b + \Delta)/N, p/N\big)$, because for exactly these i, we have $S^i x_0 < p/N \leq S^{i+1} x_0$.

Let $X = \big\{\{x_0 - ip/N\} \big| i \in \mathbb{N}, 0 \leq i < p_0 + q_0\big\}$. Put $p' = p + \Delta/k$, and let $Y = \big\{\{x_0 - ip'/N\} \big| i \in \mathbb{N}, 0 \leq i < p_0 + q_0\big\}$. We can see that $0 \leq \Delta\big((x_0 - ip/N) - (x_0 - ip'/N)\big) = i/kN < 1/N$. Thus $x_0 - ip/N \in \big[\frac{p-b+\Delta}{N}, \frac{p}{N}\big)$ if and only if

$$x_0 - ip'/N \in \begin{cases} \big(\frac{p-b}{N}, \frac{p-1}{N}\big] & \text{in the case } \Delta = +1, \\ \big[\frac{p-b-1}{N}, \frac{p}{N}\big) & \text{in the case } \Delta = -1. \end{cases} \tag{8}$$

In both cases, the length of the interval is $\frac{b-\Delta}{N}$. From $\det \mathbf{A} = \Delta$, it is easy to see that $p'/N = p_0/(p_0 + q_0)$. Because p_0 is co-prime to $p_0 + q_0$, we get

$\big\{\{ip_0/(p_0+q_0)\}\big|i \in \mathbb{N}, 0 \le i < p_0+q_0\big\} = \big\{i/(p_0+q_0)\big|i \in \mathbb{N}, 0 \le i < p_0+q_0\big\}$. But this means that the set Y is uniformly distributed on the interval $[0,1)$, which gives

$$b_0 = \#\Big(X \cap \big[\tfrac{p-b+\Delta}{N}, \tfrac{p}{N}\big)\Big) \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\},$$

where $\beta = (p_0+q_0)\frac{b-\Delta}{N}$ is number of elements of Y multiplied by the length of the interval in (8). Together we get

$$|\beta - b_0| < 1, \tag{9}$$

which is equivalent to condition (a). □

The proof of the other implication is divided into several lemmas.

Lemma 2. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$*, let* $b \in \mathbb{N}$ *with* $\frac{1+\Delta}{2} \le b \le \min\{p_0+p_1, q_0+q_1\} - \frac{1-\Delta}{2}$.

Denote $N = \|\mathbf{A}\|$ *and* $p = p_0 + p_1$ *integers,* $I = \big[\frac{p-b+\Delta}{N}, \frac{p}{N}\big)$ *an interval,* $X_k = \big\{\{k/N\}, S\{k/N\}, S^2\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}\big\}$ *a set of numbers for any* $k \in \mathbb{Z}$*, where* S *is the 2-interval exchange with the slope* $\varepsilon = p/N$*, and denote* $\beta = \frac{p_0+q_0}{N}(b - \Delta)$.

Then for all $b_0 \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$ *such that*

$$b_0 \le \min\{p_0, q_0\} \quad \text{and} \quad b - \Delta - b_0 \le \min\{p_1, q_1\}, \tag{10}$$

there exist $k', k'' \in \{0, \ldots, N-1\}$*,* $k' \ne k''$ *such that*

$$\#(X_{k'} \cap I) = \#(X_{k''} \cap I) = b_0. \tag{11}$$

Proof. Denote $r(k) = \#(X_k \cap I)$ for $k \in \mathbb{Z}$. We can see that $\sum_{k=0}^{N-1} r(k) = (b-\Delta)(p_0+q_0)$. According to (9), we know that $r(k) \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$ for all $k \in \mathbb{Z}$. Let

$$C_L = \#\big\{k \in \{0,\ldots,N-1\}\big|r(k) = \lfloor\beta\rfloor\big\},$$
$$C_U = \#\big\{k \in \{0,\ldots,N-1\}\big|r(k) = \lceil\beta\rceil\big\}.$$

We will proof the lemma by contradiction. Suppose C_L or $C_U \in \{0,1\}$. The numbers satisfy equations

$$C_L\lfloor\beta\rfloor + C_U\lceil\beta\rceil = N\beta \quad \text{and} \quad C_L + C_U = N.$$

If $C_L = 0$ or $C_U = 0$, necessarily $\beta \in \mathbb{N}$ and (11) is satisfied for all $k \in \mathbb{Z}$. Otherwise, there is a unique solution

$$C_L = N\{-\beta\} \quad \text{and} \quad C_U = N\{\beta\}.$$

Using relation $p_0N - (p_0 + p_1)(p_0 + q_0) = \Delta$, we get

$$C_U \equiv (p_0 + q_0)(b - \Delta) \pmod N$$
$$b - \Delta \equiv -\Delta(p_0 + p_1)C_U \pmod N.$$

Suppose $C_U = 1$ or $C_L = 1$, i.e. $C_U \equiv \pm 1$. Then $b = (p_0 + p_1) + \Delta$ or $b = (q_0 + q_1) + \Delta$. For $\Delta = +1$, this is in contradiction with the conditions. For $\Delta = -1$, discuss the following two cases.

1. Suppose $b = (p_0 + p_1) + \Delta$. This means $C_U = 1$. But then $b_0 = \lceil \beta \rceil = \lceil \frac{p_0 N - \Delta}{N} \rceil = p_0 + 1$ does not satisfy condition (10) of the lemma.
2. Suppose $b = (q_0 + q_1) + \Delta$. That means $C_L = 1$. But then $b_0 = \lfloor \beta \rfloor = q_0 - 1$ and $b - \Delta - b_0 = q_1 + 1$ again does not satisfy (10). □

Lemma 3. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$*, let* $b \in \mathbb{N}$ *with* $\frac{1+\Delta}{2} \leq b \leq \min\{p_0 + p_1, q_0 + q_1\} - \frac{1-\Delta}{2}$.

Denote $N = \|\mathbf{A}\|$ *and* $p = p_0 + p_1$ *integers,* $I = \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right)$ *an interval,* $X_k = \left\{\{k/N\}, S\{k/N\}, S^2\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}\right\}$ *a set of numbers for any* $k \in \mathbb{Z}$*, where* S *is the 2-interval exchange with slope* $\varepsilon = p/N$*, and denote* $\beta = \frac{p_0+q_0}{N}(b - \Delta)$*. Define morphisms* φ_k *for* $k \in \mathbb{Z}$ *in the following way:*

- *the word* $\varphi_k(0)$ *codes* $\{k/N\}, S\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}$*;*
- *the word* $\varphi_k(1)$ *codes* $S^{p_0+q_0}\{k/N\}, \ldots, S^{N-1}\{k/N\}$*.*

Let $k_0 \in \mathbb{Z}$ *such that* $\#(X_{k_0} \cap I) = \#(X_{k_0-p} \cap I)$*. Then*

$$\varphi_{k_0} \propto \varphi_{k_0+b-\Delta} \quad \text{or} \quad \varphi_{k_0-p} \propto \varphi_{k_0-p+b-\Delta},$$

and the number of B's in the ternarization of the images of the letter 0 is $\#(X_{k_0} \cap I)$*.*

Proof. Let us take the orbit

$$\{k_0/N\}, S\{k_0/N\}, \ldots, S^{p_0+q_0}\{k_0/N\}. \tag{12}$$

Let $t^{(k)}$ be a word of the length $p_0 + q_0$ that codes orbit of transformation S to the alphabet $\{0, 0', 1, 1'\}$ with a different code than (3):

$$t_i^{(k)} = \begin{cases} 0 & \text{if } S^i\{k/N\} \in \left[0, \frac{p-b+\Delta}{N}\right), \\ 0' & \text{if } S^i\{k/N\} \in \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right) = I, \\ 1 & \text{if } S^i\{k/N\} \in \left[\frac{p}{N}, \frac{N-b+\Delta}{N}\right), \\ 1' & \text{if } S^i\{k/N\} \in \left[\frac{p-b+\Delta}{N}, 1\right). \end{cases} \tag{13}$$

From definition of S, we see that $t_i^{(k)} = 0' \Leftrightarrow t_{i+1}^{(k)} = 1'$. Define two morphisms $\tau, \tau' : \{0, 0', 1, 1'\}^* \to \{0, 1\}^*$ as

$$\begin{array}{llll} \tau(0) = 0, & \tau(0') = 0, & \tau(1) = 1, & \tau(1') = 1, \\ \tau'(0) = 0, & \tau'(0') = 1, & \tau'(1) = 1, & \tau(1') = 0. \end{array}$$

If $t^{(k)}$ does not start with $1'$ and does not end with $0'$, then the word $\varphi_k(0) = \tau(t^{(k)})$ is $|t^{(k)}|_{0'}$-amicable to $\tau'(t^{(k)}) = \varphi_{k+b-\Delta}(0)$. Moreover, $|t^{(k)}|_{0'} = \#(X_k \cap I)$.

We know that $S\{k_0/N\} = \{(k_0 - p)/N\}$, which means that there exist letters $a, b \in \{0, 0', 1, 1'\}$ such that $t^{(k_0)}a = bt^{(k_0-p)}$ and $a = 0' \Leftrightarrow b = 0'$, because $|t^{(k_0)}|_{0'} = |t^{(k_0-p)}|_{0'}$. If $a = 0'$ then $t^{(k_0)}$ does not end with $0'$, because in that case $b = 1'$. If $a \neq 0'$ then $t^{(k_0-p)}$ does not start with $1'$ and does not end with $0'$. Putting these facts together with facts from the proof of Proposition 11 we get the statement. $\square$

Lemma 4. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$, *let* $b \in \mathbb{N}$ *with* $\frac{1+\Delta}{2} \leq b \leq \min\{p_0 + p_1, q_0 + q_1\} - \frac{1-\Delta}{2}$.

Denote $N = \|\mathbf{A}\|$, $q = q_0 + q_1$ *and* $p = p_0 + p_1$ *integers,* $I = \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right)$ *an interval,* $X_k = \left\{\{k/N\}, S\{k/N\}, S^2\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}\right\}$ *a set of numbers for any* $k \in \mathbb{Z}$, *where* S *is the 2-interval exchange with the slope* $\varepsilon = p/N$, *and denote* $\beta = \frac{p_0+q_0}{N}(b - \Delta)$.

Let $k_0 \in \mathbb{Z}$ *be a number such that if* $\Delta = -1$ *and* $b = \min\{p, q\} - 1$ *then*

$$k_0 \not\equiv \begin{cases} -1 \pmod N & \text{in the case } p > q, \\ p - b - 1 \pmod N & \text{in the case } p < q. \end{cases} \tag{14}$$

Then

$$\#(X_{k_0} \cap I) = \#(X_{k_0+p} \cap I) \quad \text{or} \quad \#(X_{k_0} \cap I) = \#(X_{k_0-p} \cap I).$$

Proof. Define the words $t^{(k)}$ by (13) in the same way as in the previous proof. Denote $\ell = p_0 + q_0$. Then we know that there exist letters $a_0, \ldots, a_{\ell+1} \in \{0, 0', 1, 1'\}$ such that

$$\begin{aligned} t^{(k_0+p)} &= a_0 a_1 a_2 \cdots a_{\ell-1}, \\ t^{(k_0)} &= a_1 a_2 \cdots a_{\ell-1} a_\ell, \\ t^{(k_0-p)} &= a_2 \cdots a_{\ell-1} a_\ell a_{\ell+1}. \end{aligned}$$

Remind that $\#(X_{k_0+p} \cap I) = |t^{(k)}|_{0'}$. The proof will be done by contradiction. Suppose that $|t^{(k_0+p)}|_{0'} \neq |t^{(k_0)}|_{0'} \neq |t^{(k_0-p)}|_{0'}$. There are only two possible values of these numbers, thus $|t^{(k_0+p)}|_{0'} = |t^{(k_0-p)}|_{0'}$. This together gives either $a_0 = a_{\ell+1} = 0'$ or $a_1 = a_\ell = 0'$. It means that there exist

$\xi \in I = \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right)$ and $\omega = \pm 1$ such that $S^{\ell+\omega}\xi \in I$. We can take $\xi \in \frac{1}{N}\mathbb{Z}$. Since $\ell p = p_0 N - \Delta$, we have

$$S^{\ell+\omega}\xi \equiv \xi - \frac{(\ell+\omega)p}{N} \equiv \frac{\omega p - \Delta}{N} \pmod 1.$$

This gives

$$\begin{aligned} S^{\ell+\omega}\xi - \xi &= \frac{p-\omega\Delta}{N} \\ \text{or} \quad S^{\ell+\omega}\xi - \xi &= \frac{p-\omega\Delta}{N} - 1 = -\frac{q+\omega\Delta}{N}. \end{aligned}$$

This enforces $b-1-\Delta \geq \min\{p,q\}-1$ for the interval I to be large enough to contain both ξ and $S^{\ell+\omega}\xi$.

For $\Delta = +1$, this is in contradiction with $b \leq \min\{p,q\}$.

For $\Delta = -1$ we get only one admissible $b = \min\{p,q\} - 1$. If $p = \min\{p,q\}$, it gives $\omega = -1$ and $\xi = \frac{p-b-1}{N}$, which implies $k_0 \equiv p-b-1 \pmod N$. If $q = \min\{p,q\}$, it gives $\omega = +1$ and $\xi = \frac{p-1}{N}$, which implies $k_0 \equiv -1 \pmod N$. Both these cases are in contradiction with (14). □

Proof of the implication ($\Leftarrow$). From [3], the incidence matrix of the ternarization $\mathrm{ter}(\varphi,\psi)$ is fully described by the matrix $\mathbf{A}$ and numbers b_0 and $b = b_0 + b_1 + \Delta$. The condition (a) is equivalent to (9) and it gives at most two values of b_0. If $\beta \in \mathbb{N}$, there is nothing to do as we have at least one pair of b-amicable morphisms $\varphi \propto \psi$ for $\mathbf{A}$, and its incidence matrix satisfies all three conditions.

For $\beta \notin \mathbb{N}$, we want to show that for both $b_0 \in \{\lfloor\beta\rfloor, \lceil\beta\rceil\}$ there exist $\varphi \propto \psi$ with $|\mathrm{ter}(\varphi(0),\psi(0))|_B = b_0$. Because the elements of the matrix $\mathbf{B}$ are non-negative, the condition (10) of Lemma 2 is satisfied and we have two different k', k''. At least one of them satisfies (14). Lemma 4 then provides k_0 satisfying the conditions of Lemma 3 that gives a pair of amicable Sturmian morphisms, ternarization of which has the incidence matrix $\mathbf{B}$. □

4 Conclusions and open problems

Matrices of 3iet-preserving morphisms were studied in [3]. The authors give a necessary condition on $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to be an incidence matrix of a 3iet-preserving morphism:

$$\mathbf{B}\mathbf{E}\mathbf{B}^{\mathsf{T}} = \pm\mathbf{E}, \quad \text{where} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

However, this condition is not sufficient. In our contribution, we study 3iet-preserving morphisms $\eta = \mathrm{ter}(\varphi,\psi)$ arising from pairs of amicable Sturmian morphisms $\varphi \propto \psi$. Our Theorem 12 gives sufficient and necessary condition for any matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to satisfy $\mathbf{B} = \mathbf{M}_\eta$ for some ternarization $\eta = \mathrm{ter}(\varphi, \psi)$.

It remains to answer the question about the role of the monoid

$$\mathcal{M}_{\mathrm{ter}} = \big\{\mathrm{ter}(\varphi, \psi) \big| \varphi, \psi \text{ amicable morphisms}\big\}$$

in the whole monoid $\mathcal{M}_{3\mathrm{iet}}$ of all 3iet-preserving morphisms.

It seems that using similar proof as [2] for Theorem 5 we can proof the following statement.

Conjecture. *Let $\eta \in \mathcal{M}_{3\mathrm{iet}}$. Then there exists $i \in \{0,1,2,3\}$ such that $\eta \circ \xi_i \in \mathcal{M}_{\mathrm{ter}}$, where $\xi_0, \ldots, \xi_3$ are 3iet-preserving morphisms,*

$$\begin{array}{llll} \xi_0(A) = A, & \xi_1(A) = C, & \xi_2(A) = B, & \xi_3(A) = B, \\ \xi_0(B) = B, & \xi_1(B) = B, & \xi_2(B) = ACA, & \xi_3(B) = CAC, \\ \xi_0(C) = C, & \xi_1(C) = A, & \xi_2(C) = A, & \xi_3(C) = C. \end{array}$$

References

[1] P. Ambrož, A. Frid, Z. Masáková, and E. Pelantová, *On the number of factors in codings of three interval exchange*, Preprint 2009, arXiv:0904.2258v1.

[2] P. Ambrož, Z. Masáková, and E. Pelantová, *Morphisms fixing words associated with exchange of three intervals*, RAIRO Theor. Inform. Appl. **44** (2010), 3–17.

[3] P. Ambrož, Z. Masáková, and E. Pelantová, *Matrices of 3-iet preserving morphisms*, Theoret. Comput. Sci. **400** (2008), no. 1-3, 113–136.

[4] P. Arnoux, V. Berthé, Z. Masáková, and E. Pelantová, *Sturm numbers and substitution invariance of 3iet words*, Integers **8** (2008), A14, 17.

[5] J. Berstel, *Recent results in Sturmian words*, Developments in language theory, II (Magdeburg, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 13–24.

[6] J. Berstel and P. Séébold, *Morphismes de sturm*, Bull. Belg. Math. Soc. **1** (1994), 175–189.

[7] J. Cassaigne, *Sequences with grouped factors*, Developments in language theory III, Aristotle University of Thessaloniki, Greece, 1998, pp. 211–222.

[8] S. Ferenczi, C. Holton, and L. Q. Zamboni, *Structure of three-interval exchange transformations. II. A combinatorial description of the trajectories*, J. Anal. Math. **89** (2003), 239–276.

[9] L. Háková, *Morphisms on generalized sturmian words*, Master's thesis, Czech Technical University in Prague, 2008.

[10] T. Hejda, *Amicable morphisms on sturmian words*, Bachelor's thesis, Czech Technical University in Prague, 2010.

[11] A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 81–106.

[12] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002.

[13] P. Séébold, *On the conjugation of standard morphisms*, Theoret. Comput. Sci. **195** (1998), no. 1, 91–109.