Morphisms preserving the set of words coding three interval exchange

Tomáš Hejda

Department of Mathematics FNSPE
Czech Technical University in Prague
Czech Republic
e-mail: tohe@centrum.cz

Abstract

Any amicable pair φ, ψ of Sturmian morphisms enables a construction of a ternary morphism η which preserves the set of infinite words coding 3-interval exchange. We determine the number of amicable pairs with the same incidence matrix in $SL(2,\mathbb{N})$ and we study incidence matrices associated with the corresponding ternary morphisms η.

1 Introduction

Sturmian words are well-described objects in combinatorics on words. They can be defined in several equivalent ways [5], e.g. as words coding a two-interval exchange transformation with irrational ratio of lengths of the intervals. Morphisms preserving the set of Sturmian words are called *Sturmian* and they form a monoid generated by three of its elements (see [6, 11]). Let us denote this monoid by $\mathcal{M}_{\text{Sturm}}$.

In this paper, we consider morphisms preserving the set of words coding a three-interval exchange transformation with permutation $(3,2,1)$, the so-called *3iet words*. We call these morphisms *3iet-preserving*. Monoid of these morphisms, denoted by $\mathcal{M}_{\text{3iet}}$, is not fully described. It is shown (see [9]) that the monoid $\mathcal{M}_{\text{3iet}}$ is not finitely generated. Recently, in [2], pairs of amicable Sturmian morphisms were defined. The authors used this notion to describe morphisms that have as a fixed point a non-degenerate 3iet word, i.e., 3iet word with complexity $\mathcal{C}(n) = 2n+1$. Using the operation of "ternarization", we can assign a morphism $\eta = \operatorname{ter}(\varphi,\psi)$ over a ternary alphabet to a pair of amicable Sturmian morphisms. We show that such η is a 3iet-preserving morphism. Moreover, we show that the set

$$\mathcal{M}_{\text{ter}} = \big\{\operatorname{ter}(\varphi,\psi)\big|\varphi,\psi \text{ amicable Sturmian morphisms}\big\} \qquad (1)$$

is a monoid, but it does not cover the whole monoid $\mathcal{M}_{3\text{iet}}$.

We also study the incidence matrices of morphisms $\eta \in \mathcal{M}_{\text{ter}}$. From the definition of amicable Sturmian morphisms φ, ψ we can derive that φ and ψ have the same incidence matrix $\mathbf{A} \in \mathbb{N}^{2\times 2}$, where $\det \mathbf{A} = \pm 1$. As shown in [13], for every matrix $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ with $\det \mathbf{A} = \pm 1$, there exist $p_0 + p_1 + q_0 + p_1 - 1$ Sturmian morphisms with incidence matrix $\mathbf{A}$. We will show the following theorem concerning the number of pairs of amicable Sturmian morphisms with a given matrix.

Theorem 1. *Let $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ be a matrix with $\det \mathbf{A} = \pm 1$. Then there exist exactly*

$$m\big(\|\mathbf{A}\| - 1\big) + \frac{m}{2}\big(\det \mathbf{A} - m\big) \tag{2}$$

pairs of amicable Sturmian morphisms with incidence matrix $\mathbf{A}$, where $m = \min\{p_0 + p_1, q_0 + q_1\}$ and $\|\mathbf{A}\| = p_0 + p_1 + q_0 + q_1$.

Moreover, for such a given matrix $\mathbf{A}$, we will describe all matrices $\mathbf{B} \in \mathbb{N}^{3\times 3}$ such that $\mathbf{B}$ is an incidence matrix of $\eta = \text{ter}(\varphi, \psi)$ for amicable Sturmian morphisms φ, ψ with incidence matrix $\mathbf{A}$.

2 Preliminaries

Interval exchange. We consider Sturmian words, i.e., aperiodic infinite words given by exchange of 2 intervals with permutation $(2, 1)$, and words given by exchange of 3 intervals with permutation $(3, 2, 1)$. Let us recall that general r-interval exchange transformations were introduced already in [10]. The 2-interval exchange transformation $S : [0, 1) \to [0, 1)$ is determined by its slope $\varepsilon \in (0, 1)$ and is given by

$$Sx = \begin{cases} x + 1 - \varepsilon & \text{if } x \in [0, \varepsilon) \\ x - \varepsilon & \text{if } x \in [\varepsilon, 1). \end{cases}$$

The orbit of a point $x_0 \in [0, 1)$ with respect to the transformation S, i.e., the sequence $x_0, Sx_0, S^2x_0, \ldots$ can be coded by an infinite word $u = (u_i)_{i=0}^{\infty}$ over the binary alphabet $\{0, 1\}$. The infinite word is given by

$$u_i = \begin{cases} 0 & \text{if } S^i x \in [0, \varepsilon), \\ 1 & \text{if } S^i x \in [\varepsilon, 1). \end{cases}$$

It is a well-known fact that for an irrational ε, the word u is Sturmian. Using the same construction on the partition of the interval $(0, 1]$ into $(0, \varepsilon] \cup (\varepsilon, 1]$, we again obtain a Sturmian word. On the other hand, every Sturmian word can be obtained by one of the above two constructions. The set of Sturmian words will be denoted by $\mathcal{W}_{\text{Sturm}}$.

The 3-interval exchange transformation T is determined by two parameters $\alpha, \beta \in (0,1)$ satisfying $\alpha + \beta < 1$. Using parameters α, β and $\gamma = 1-\alpha-\beta$ we partition the interval $[0,1)$ into $I_A = [0,\alpha)$, $I_B = [\alpha, \alpha+\beta)$ and $I_C = [\alpha+\beta, 1)$. The mapping T is given by

$$Tx = \begin{cases} x+\beta+\gamma & \text{if } x \in I_A, \\ x-\alpha+\gamma & \text{if } x \in I_B, \\ x-\alpha-\beta & \text{if } x \in I_C. \end{cases}$$

The orbit of a point $x_0 \in [0,1)$ with respect to the transformation T is coded by a word $u = (u_i)_{i=0}^{\infty}$ over the ternary alphabet $\{A, B, C\}$:

$$u_i = X \quad \text{if} \quad T^i x_0 \in I_X.$$

Similarly to the case of 2-interval exchange transformation, we can define the exchange of 3 intervals using the partition $(0,1] = (0,\alpha] \cup (\alpha, \alpha+\beta] \cup (\alpha+\beta, 1]$. If $\frac{1-\alpha}{1+\beta}$ is irrational, the infinite word u is aperiodic, and we call it a *3iet word*; the set of these words is denoted by $\mathcal{W}_{\text{3iet}}$. For combinatorial properties of 3iet words, see [8].

Words over a finite alphabet. Besides the infinite words, we consider *finite words* over the alphabet $\mathcal{A}$. We write $w = w_0 w_1 \cdots w_{n-1}$, where $w_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, $i < n$. We denote by $|w|$ the length n of the finite word w. We denote by $|w|_a$ the number of occurrences of a letter $a \in \mathcal{A}$ in the word w. The set of all finite words over the alphabet $\mathcal{A}$ including the empty word ϵ is denoted by $\mathcal{A}^*$. The set $\mathcal{A}^*$ with the operation of concatenation is a monoid. On the set $\mathcal{A}^*$ we define a relation of *conjugation*: $w \sim w'$, if there exists $v \in \mathcal{A}^*$ such that $wv = vw'$. A *morphism* from $\mathcal{A}^*$ to $\mathcal{B}^*$ is a mapping $\varphi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. It is clear that a morphism is well defined by images of letters $\varphi(a)$ for all $a \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}$, then φ is called a *morphism over* $\mathcal{A}$.

The set of *infinite words* over the alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$. The action of a morphism can be naturally extended to an infinite word $(u_i)_{i\in\mathbb{N}}$ putting $\varphi(u) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$. If an infinite word $u \in \mathcal{A}^{\mathbb{N}}$ satisfies $\varphi(u) = u$, we call it a *fixed point* of the morphism φ over $\mathcal{A}$.

To a morphism φ over $\mathcal{A}$ we assign an *incidence matrix* $\mathbf{M}_\varphi$ defined by $(\mathbf{M}_\varphi)_{ab} = |\varphi(a)|_b$ for all $a, b \in \mathcal{A}$.

The *language* of an infinite word u is the set of all its factors. Let us recall that a finite word $w \in \mathcal{A}^*$ is a *factor* of $u = (u_i)_{i\in\mathbb{N}}$, if there exist indices $n, j \in \mathbb{N}$ such that $w = u_n u_{n+1} \cdots u_{n+j-1}$. The language of an infinite word is denoted by $\mathcal{L}(u)$.

It is known that the language of neither Sturmian nor 3iet word depends on the point $x_0 \in [0,1)$, the orbit of which the infinite word codes. It depends only on slope ε or parameters α, β.

The *(factor) complexity* of an infinite word u is a mapping $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$, which returns the number of factors of u of the length n, thus $\mathcal{C}_u(n) = \#\{w \in \mathcal{L}(u) \big| |w| = n\}$. It is shown [12] that any aperiodic word u satisfies $\mathcal{C}_u(n) \geq n+1$ for all $n \in \mathbb{N}$. Aperiodic words with minimal complexity, i.e., $\mathcal{C}_u(n) = n+1$, are exactly the Sturmian words. Aperiodic words coding 3-interval exchange transformations, called here 3iet words, have the complexity $\mathcal{C}_u(n) \leq 2n+1$ for all $n \in \mathbb{N}$. If a 3iet word $u \in \mathcal{W}_{3\text{iet}}$ satisfies $\mathcal{C}_u(n) = 2n+1$ for all $n \in \mathbb{N}$, we call it a *non-degenerate* 3iet word; otherwise we call it a *degenerate* 3iet word and it is a quasi-Sturmian word (see [7]).

Amicable words and morphisms. In the article [4], authors show the close connection between 3iet and Sturmian words using two morphisms $\sigma_{01}, \sigma_{10} : \{A,B,C\}^* \to \{0,1\}^*$ given by

$$\begin{aligned} \sigma_{01}(A) &= 0, & \sigma_{10}(A) &= 0, \\ \sigma_{01}(B) &= 01, & \sigma_{10}(B) &= 10, \\ \sigma_{01}(C) &= 1, & \sigma_{10}(C) &= 1. \end{aligned}$$

In [4], the following theorem is proved.

Theorem 2. *An infinite ternary word $u \in \{A,B,C\}^{\mathbb{N}}$ is a 3iet word if and only if the words $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian.*

This theorem motivated the authors of [1] to introduce the relation of amicability of words.

Definition 3. Let $w, w' \in \{0,1\}^*$, let $b \in \mathbb{N}$. We say that w is *b-amicable* to w', if there exists a factor $v \in \{A,B,C\}^*$ of some 3iet word such that

$$w = \sigma_{01}(v), \qquad w' = \sigma_{10}(v) \quad \text{and} \quad |v|_B = b.$$

We say that w is *amicable* to w', if w is b-amicable to w' for some $b \in \mathbb{N}$, and we denote it by $w \propto w'$.

The ternary word v is called a *ternarization* of w and w', and we write $v = \operatorname{ter}(w, w')$.

It is easy to see that if $w \propto w'$, then they are factors of the same Sturmian word and the numbers of occurrences of 0's and 1's in w and w' coincide.

In [1], the notion of "amicable words" plays a crucial role in enumeration of words with length n occurring in a 3iet word. In [2], the authors investigate ternary morphisms that have a non-degenerate 3iet fixed point using the following notion of amicability of two Sturmian morphisms.

Definition 4. Let φ, ψ be Sturmian morphisms over the alphabet $\{0,1\}$. We say that φ is *amicable* to ψ, if

$$\begin{aligned} \varphi(0) &\propto \psi(0), \\ \varphi(01) &\propto \psi(10) \\ \text{and} \quad \varphi(1) &\propto \psi(1). \end{aligned}$$

We denote this relation by $\varphi \propto \psi$. The morphism η over the ternary alphabet $\{A, B, C\}$, given by

$$\begin{aligned} \eta(A) &= \operatorname{ter}(\varphi(0), \psi(0)), \\ \eta(B) &= \operatorname{ter}(\varphi(01), \psi(10)), \\ \eta(C) &= \operatorname{ter}(\varphi(1), \psi(1)) \end{aligned}$$

is called the *ternarization* of morphisms φ and ψ, and is denoted by $\eta = \operatorname{ter}(\varphi, \psi)$. Set of these η is denoted by $\mathcal{M}_{\mathrm{ter}}$.

The article [2] states the following theorem:

Theorem 5. *Let η be a ternary morphism with non-degenerate 3iet fixed point. Then $\eta \in \mathcal{M}_{\mathrm{ter}}$ or $\eta^2 \in \mathcal{M}_{\mathrm{ter}}$.*

3 Main results

Analogously to the terminology introduced for Sturmian words and morphisms in [6], the ternarization η, having a 3iet fixed point, is *locally 3iet-preserving*, i.e. there exists $u \in \mathcal{W}_{3\mathrm{iet}}$ such that $\eta(u) \in \mathcal{W}_{3\mathrm{iet}}$. We now prove a partial result about *(globally) 3iet-preserving* morphisms, i.e., ternary morphisms η such that

$$\eta(u) \in \mathcal{W}_{3\mathrm{iet}} \quad \text{for all} \quad u \in \mathcal{W}_{3\mathrm{iet}}.$$

Proposition 6. *Let $\eta = \operatorname{ter}(\varphi, \psi)$ for amicable Sturmian morphisms $\varphi \propto \psi$. Then η is a globally 3iet-preserving morphism.*

Proof. Directly from definitions we see that

$$\begin{aligned} &\sigma_{01}\eta(A) = \varphi(0), && \sigma_{01}\eta(B) = \varphi(01), && \sigma_{01}\eta(C) = \varphi(1), \\ &\sigma_{10}\eta(A) = \psi(0), && \sigma_{10}\eta(B) = \psi(10), && \sigma_{10}\eta(C) = \psi(1). \end{aligned}$$

Therefore

$$\sigma_{01}\eta(v) = \varphi\sigma_{01}(v) \qquad \text{and} \qquad \sigma_{10}\eta(v) = \psi\sigma_{10}(v) \tag{3}$$

for any factor v of a 3iet word $u \in \mathcal{W}_{3\mathrm{iet}}$. According to Theorem 2 we get that $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian words, and since φ and ψ are Sturmian morphisms, we obtain that $\sigma_{01}\eta(u)$ and $\sigma_{10}\eta(u)$ are Sturmian words as well. $\square$

Proposition 7. *Let $\varphi_i \propto \psi_i$ be Sturmian morphisms, for $i = 1, 2$. Then*

$$\operatorname{ter}(\varphi_1, \psi_1) \circ \operatorname{ter}(\varphi_2, \psi_2) = \operatorname{ter}(\varphi_1 \circ \varphi_2, \psi_1 \circ \psi_2).$$

Proof. It can be shown that the relation of amicability is preserved by composition of morphisms. More precisely $\varphi_1\varphi_2 \propto \psi_1\psi_2$. Denote $\eta_1 = \operatorname{ter}(\varphi_1, \psi_1)$, $\eta_2 = \operatorname{ter}(\varphi_2, \psi_2)$. Using the relation (3), we see that for all $v \in \{A, B, C\}^*$

$$\begin{aligned} &\sigma_{01}\eta_1\eta_2(v) = \varphi_1\sigma_{01}\eta_2(v) = \varphi_1\varphi_2\sigma_{01}(v) \\ \text{and} \quad &\sigma_{10}\eta_1\eta_2(v) = \psi_1\sigma_{10}\eta_2(v) = \psi_1\psi_2\sigma_{10}(v). \end{aligned}$$

But this means that $\eta_1\eta_2 = \operatorname{ter}(\varphi_1\varphi_2, \psi_1\psi_2)$. □

As a consequence of previous two propositions, we can state the following theorem.

Theorem 8. *The set $\mathcal{M}_{\text{ter}}$ of all ternarizations of amicable Sturmian morphisms with the operation of composition of morphisms is a sub-monoid of the monoid $\mathcal{M}_{\text{3iet}}$ of all globally 3iet-preserving morphisms.*

Unfortunately, $\mathcal{M}_{\text{ter}} \subsetneq \mathcal{M}_{\text{3iet}}$. Consider for example the morphism

$$\eta(A) = B, \qquad \eta(B) = CAC, \qquad \eta(C) = C.$$

As shown in [9], this morphism is 3iet-preserving, but it can be easily verified that it is not a ternarization of any pair of Sturmian morphisms. Another even simpler example is the morphism interchanging the letters A and C, which is clearly 3iet-preserving.

Now, our goal is to determine the number of amicable pairs of morphisms with incidence matrix $\mathbf{A}$ of $\det \mathbf{A} = \pm 1$. We use the notion of b-amicable morphisms.

Definition 9. Let φ and ψ be binary morphisms and let $b \in \mathbb{N}$. We say that φ is *b-amicable* to ψ, if φ is amicable to ψ and the number of occurrences of B in $\operatorname{ter}(\varphi(01), \psi(10))$ is b.

Proposition 10. *Let $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ be a matrix with $\det \mathbf{A} = \pm 1$ and $b \in \mathbb{N}$. Put $p = p_0 + p_1$, $q = q_0 + q_1$. Then the number $c_{\mathbf{A}}(b)$ of pairs of b-amicable morphisms with matrix $\mathbf{A}$ is equal to*

$$c_{\mathbf{A}}(b) = \begin{cases} \|\mathbf{A}\| - b & \text{if } \det \mathbf{A} = +1 \text{ and } 1 \le b \le \min\{p, q\}, \\ \|\mathbf{A}\| - b - 2 & \text{if } \det \mathbf{A} = -1 \text{ and } 0 \le b \le \min\{p, q\} - 1, \\ 0 & \text{otherwise.}^1 \end{cases}$$

[1]Let us recall that $\left\| \begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix} \right\| = p_0 + q_0 + p_1 + q_1$.

Sketch of the proof. Denote $N = \|\mathbf{A}\|$. Let us consider a Sturmian morphism φ with incidence matrix $\mathbf{A}$, and let $v = \varphi(01)$. Then $|v| = N$ and $|v|_0 = p$. The word v is a factor of some Sturmian word, nevertheless v codes a 2-interval exchange transformation S with rational slope $\varepsilon = p/N$. All conjugates to v — there are N such words including v itself — arise from the same transformation coding the sequence

$$x_0, Sx_0, S^2x_0, \ldots, S^{N-1}x_0 \qquad \text{for } x_0 = \frac{0}{N}, \frac{1}{N}, \ldots, \frac{N-1}{N}. \tag{4}$$

Consider now a matrix $\mathbf{A}$ with $\det \mathbf{A} = +1$. For every such matrix, there exist $N-1$ Sturmian morphisms $\varphi_1, \ldots, \varphi_{N-1}$. One of these morphisms, the so-called standard morphism φ_{std}, satisfies that there exists $w \in \{0,1\}^*$ such that $\varphi_{\text{std}}(01) = w01$ and $\varphi_{\text{std}}(10) = w10$, which means that φ_{std} is 1-amicable to itself (for details, see [13]).

For all morphisms φ_i, $1 \le i \le N-1$, the word $\varphi_i(01)$ is conjugate to $\varphi_{\text{std}}(01)$. From this, it can be shown that each of these morphisms is 1-amicable to itself. Each of these words $\varphi_i(01)$ codes a sequence (4) with $x_0 = k_i/N$ where $0 \le k_i \le N-2$. Using similar tricks as in [1] we can prove that $\varphi_i(01)$ is b-amicable to $\varphi_j(10)$ if and only if $0 \le k_j - k_i = b - 1 \le \min\{p,q\} - 1$.

Combining all these facts the theorem can be proven for $\det \mathbf{A} = +1$. The proof for $\det \mathbf{A} = -1$ would be done in a very similar way. □

Proof of Theorem 1. The formula (2) can be obtained by summation of numbers $c_{\mathbf{A}}(b)$ from the previous proposition. □

To each pair of amicable Sturmian morphisms, an incidence matrix of its ternarization is assigned. We now fully describe which matrices from $\mathbb{N}^{3\times3}$ are matrices of ternarizations.

Theorem 11. *A matrix $\mathbf{B} \in \mathbb{N}^{3\times3}$ is the incidence matrix of the ternarization of a pair of amicable Sturmian morphisms if and only if there exist matrix $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times2}$ with $\det \mathbf{A} = \Delta = \pm1$ and numbers $b_0, b_1 \in \mathbb{N}$ such that*

(a) $\left|\frac{b_0(p_1+q_1)-b_1(p_0+q_0)}{p_0+q_0+p_1+q_1}\right| < 1$,

(b) $\frac{1-\Delta}{2} \le b_0 + b_1 \le \min\{p_0+p_1, q_0+q_1\} - \frac{\Delta+1}{2}$,

(c) $\mathbf{B} = \mathbf{P}\begin{pmatrix} \mathbf{A} & \begin{smallmatrix} b_0 \\ b_1 \end{smallmatrix} \\ 0\ 0 & \Delta \end{pmatrix}\mathbf{P}^{-1}$, *where* $\mathbf{P} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

Sketch of the proof. Let us denote $p = p_0 + p_1$, $q = q_0 + q_1$, $N = p + q$ and $b = b_0 + b_1 + \Delta$. Then we can see that condition (c) gives

$$\mathbf{B} = \begin{pmatrix} p_0 - b_0 & b_0 & q_0 - b_0 \\ p - b & b & q - b \\ p_1 - b_1 & b_1 & q_1 - b_1 \end{pmatrix}. \tag{5}$$

We will sketch the proof for $\Delta = +1$ as the other case is very similar. Let us start with the implication $\Rightarrow$.

The fact that (c) is necessary for $\mathbf{B}$ to be an incidence matrix of a ternarization is shown in [3]. Condition (b) is necessary according to Proposition 10, so we only need to show that (a) is satisfied for the matrix of the ternarization $\eta = \mathrm{ter}(\varphi, \psi)$ of a pair of amicable Sturmian morphisms $\varphi \propto \psi$.

We can see that $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ is necessarily an incidence matrix of both φ and ψ. Let S be a 2-interval exchange transformation with rational slope $\varepsilon = p/N$. Then there exist $k, \bar{k} \in \{0, \dots, N-2\}$ such that $\varphi(01)$, $\psi(01)$ code transformation S with start points $x_0 = k/N$, $\bar{x}_0 = \bar{k}/N$; moreover, $\bar{k} - k = b - 1 = b - \Delta$. We need to determine the value of $b_0 = |\mathrm{ter}(\varphi(0), \psi(0))|_B$. The number b_0 is equal to the number of indices $i \in \{0, 1, \dots, p_0 + q_0 - 1\}$ such that $S^i x_0 \in \big[(p - b + 1)/N, p/N\big)$.

Denote by $\{x\} = x - \lfloor x \rfloor$ the fractional part of $x \in \mathbb{R}$. Then we have $S^i x_0 = \{x_0 - ip/N\}$. Denoting $X = \big\{\{x_0 - ip/N\} \big| i \in \mathbb{N}, 0 \leq i < p_0 + q_0\big\}$, we can show that

$$b_0 = \#\Big(X \cap \big[\tfrac{p-b+1}{N}, \tfrac{p}{N}\big)\Big) \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\},$$

where $\beta = \frac{p_0+q_0}{N}(b-1)$. This means that

$$|\beta - b_0| < 1, \tag{6}$$

which implies condition (a).

Let us now sketch the proof of the other implication $\Leftarrow$. From [3], the incidence matrix of the ternarization $\mathrm{ter}(\varphi, \psi)$ is fully described by the matrix $\mathbf{A}$ and numbers b_0 and $b = b_0 + b_1 + \Delta$. Let us fix a matrix $\mathbf{A}$ and $1 \leq b \leq \min\{p, q\} - 1$. The condition (a) is equivalent to (6) and it gives at most two values of b_0. If $\beta \in \mathbb{N}$, there is nothing to do as we have at least one pair of b-amicable morphisms $\varphi \propto \psi$ for $\mathbf{A}$, and its incidence matrix satisfies all three conditions.

For $\beta \notin \mathbb{N}$, we want to show that for both $b_0 \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$ there exists $\varphi \propto \psi$ with $b_0 = |\mathrm{ter}(\varphi(0), \psi(0))|_B$. The demonstration needs several statements; their proofs are too technical to be included in this extended abstract.

1. Let $X_k = \big\{\{k/N\}, S\{k/N\}, S^2\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}\big\}$ for any $k \in \mathbb{Z}$ and let $I = \big[\frac{p-b+1}{N}, \frac{p}{N}\big)$ be an interval. For both $b_0 \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$, there exist $k_1, k_2 \in \mathbb{Z}$ such that
$$\#(X_{k_1} \cap I) = \#(X_{k_2} \cap I) = b_0 \qquad \text{and} \qquad k_1 \not\equiv k_2 \pmod{N}.$$
2. Define morphisms φ_k for $k \in \mathbb{Z}$ in the following way. The word $\varphi_k(0)$ codes $\{k/N\}, S\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}$ and the word $\varphi_k(1)$ codes $S^{p_0+q_0}\{k/N\}, \ldots, S^{N-1}\{k/N\}$. Let $k_0 \in \mathbb{Z}$ such that $\#(X_{k_0} \cap I) = \#(X_{k_0+p} \cap I)$. Then
$$\varphi_{k_0} \propto \varphi_{k_0+b-1} \quad \text{or} \quad \varphi_{k_0+p} \propto \varphi_{k_0+p+b-1}.$$
3. It remains to show that for both $b_0 \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$, there exists k_0 satisfying $\#(X_{k_0} \cap I) = \#(X_{k_0+p} \cap I)$. □

4 Conclusions

Matrices of 3iet-preserving morphisms were studied in [3]. The authors give a necessary condition on $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to be an incidence matrix of a 3iet-preserving morphism:
$$\mathbf{B}\mathbf{E}\mathbf{B}^{\mathsf{T}} = \pm\mathbf{E}, \quad \text{where} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$
However, this condition is not sufficient. In our contribution, we study 3iet-preserving morphisms $\eta = \mathrm{ter}(\varphi, \psi)$ arising from pairs of amicable Sturmian morphisms $\varphi \propto \psi$. Our Theorem 11 gives sufficient and necessary condition for any matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to satisfy $\mathbf{B} = \mathbf{M}_\eta$ for some ternarization $\eta = \mathrm{ter}(\varphi, \psi)$.

It remains to answer the question about the role of the monoid
$$\mathcal{M}_{\mathrm{ter}} = \big\{\mathrm{ter}(\varphi, \psi) \big| \varphi, \psi \text{ amicable morphisms}\big\}$$
in the whole monoid $\mathcal{M}_{\mathrm{3iet}}$ of all 3iet-preserving morphisms.

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