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Obor: Matematické inženýrství
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Spřátelené morfismy na sturmovských slovech

Amicable Morphisms on Sturmian Words

BAKALÁŘSKÁ PRÁCE

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Prohlášení

Prohlašuji, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškerou použitou literaturu.

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Poděkování

Na tomto místě bych chtěl především poděkovat paní profesorce Editě Pelantové za to, že mne celý rok trpělivě vedla úskalími a problémy, které téma bakalářské práce skýtá.

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Tomáš Hejda

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Abstrakt:

Sturmovská slova a morfismy přitahují pozornost matematiků, protože je lze ekvivalentně definovat mnoha způsoby. Jedna z definic je založena na výměně dvou intervalů, slova potom nazýváme 2iet slova. Podobně lze pomocí výměny tří intervalů definovat 3iet slova. Při popisu morfismů, které mají 3iet slova za svoje pevné body, byla zjištěna úzká souvislost těchto morfismů s páry spřátelených sturmovských morfismů. Hlavním výsledkem práce je určení počtu párů spřátelených sutrmovských morfismů, které mají stejnou incidenční matici.

Klíčová slova: sturmovská slova, sturmovské morfismy, relace spřátelenosti, výměna intervalů

Title: **Amicable Morphisms on Sturmian Words**
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Abstract:

Sturmian words and morphisms grab attention of mathematicians because of their numerous equivalent definitions. One of the definitions is based on two interval exchange transformation; words are then called 2iet words. Similarly 3iet words can be defined using three interval exchange transformation. There is a close connection between these morphisms and pairs of amicable sturmian morphisms. The main result is that we found number of pairs of amicable sturmian morphisms with the same incidence matrix.

Keywords: sturmian words, sturmian morphisms, relation of amicability, interval exchange transformation

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Symbol	Description
$\mathbb{N}$	set of natural numbers, $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	set of integer numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\widehat{n}$	$\{0, 1, 2, \ldots, n-1\}$, special cases $\widehat{0} \coloneqq \emptyset$ and $\widehat{\infty} \coloneqq \mathbb{N}$
(a, b)	open interval, $\{x \in \mathbb{R} \mid a < x < b\}$
$[a, b)$	semi-closed interval, $\{x \in \mathbb{R} \mid a \leq x < b\}$
$[a, b]$	closed interval, $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
$\# B$	number of elements of the set B
$\mathbb{A}$	alphabet
$\mathbb{A}^*$	set of finite words on the alphabet $\mathbb{A}$
$\boldsymbol{\varepsilon}$	empty word in $\mathbb{A}^*$
$\mathbb{A}^+$	set of finite non-empty words on $\mathbb{A}$, $\mathbb{A}^* \setminus \{\boldsymbol{\varepsilon}\}$
$\mathbb{A}^n$	set of finite words on $\mathbb{A}$ of length n
$\mathbb{A}^{\mathbb{N}}$	set of infinite words on $\mathbb{A}$
$\mathbb{A}^{\infty}$	set of finite and infinite words on $\mathbb{A}$, $\mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$
$\lvert\boldsymbol{u}\rvert$	length of the word $\boldsymbol{u}$
$\lvert\boldsymbol{u}\rvert_a$	number of occurrences of a in $\boldsymbol{u}$
$\mathrm{Fact}(\boldsymbol{u})$	set of all factors of the word $\boldsymbol{u}$
$\mathrm{Fact}_n(\boldsymbol{u})$	set of all factors of the word $\boldsymbol{u}$ of length n
$\boldsymbol{u}^{\mathsf{R}}$	reverse word to $\boldsymbol{u}$
$\boldsymbol{u} \propto \boldsymbol{v}$, $\varphi \propto \psi$	relation of amicability of words and morphisms
$\mathrm{ter}(\cdot, \cdot)$	ternarization of words or morphisms
$\boldsymbol{u} \prec \boldsymbol{v}$	lexicographical order of words
T_α	2-interval exchange transformation
$T_{\alpha,\beta}$	3-interval exchange transformation
$\mathcal{L}(\alpha)$	language of 2-interval exchange transformation T_α

Chapter 1

Introduction

The objective of this thesis is to expand knowledge on the pairs of amicable sturmian morphisms. The article [1] shows close connection between these morphisms and 3iet preserving morphisms, specifically by the following theorem.

Theorem. *Let η be a primitive substitution fixing a non-degenerate 3iet word $\boldsymbol{u}$. Then there exist Sturmian morphisms φ and ψ having fixed points, such that $\varphi \propto \psi$ and η or η^2 is equal to* $\mathrm{ter}(\varphi, \psi)$.

In chapter two, the basic notions are defined, particularly words and morphisms, and summary of their most important properties is done. Chapter three summarizes definitions and properties of sturmian words and morphisms. Definition and properties of the relation of amicability are written in chapter four.

The new results on number of amicable sturmian morphisms are in chapter five.

The properties of non-negative integer 2×2 matrices with determinant equal to $+1$, which are widely used in the thesis, are recalled in the appendix.

Chapter 2

Words and Morphisms

2.1 Words on finite alphabets

At first, we need to define basic notions, alphabet, letter, word, etc.

Definition 2.1. **(1)** An **alphabet** (usually denoted $\mathbb{A}$) is any finite set, elements of $\mathbb{A}$ are called **letters**.

In our thesis, the alphabet will usually be the set $\{0, 1\}$ or the set $\{0, 1, 2\}$.

(2) Let $n \in \mathbb{N}$. The mapping $\boldsymbol{w} : \widehat{n} \to \mathbb{A}$ is a **finite word** over $\mathbb{A}$. Number n is called the **length** of finite word $\boldsymbol{w}$, denoted $|\boldsymbol{w}|$.

(3) $\boldsymbol{u} : \mathbb{N} \to \mathbb{A}$ is an **infinite word** over $\mathbb{A}$. We put length of infinite word as $|\boldsymbol{u}| := +\infty$.

(4) Value of the word $\boldsymbol{u}$ (as a mapping) at non-negative integer i is denoted as u_i.

(5) $\mathbb{A}^*$ is set of all finite words over $\mathbb{A}$.

(6) $\mathbb{A}^{\mathbb{N}}$ is set of all infinite words over $\mathbb{A}$.

(7) Let $n \in \mathbb{N}$. We denote $\mathbb{A}^n$ the set of all words on $\mathbb{A}$ of length n, $\mathbb{A}^n := \left\{\boldsymbol{u} \in \mathbb{A}^* \middle| |\boldsymbol{u}| = n\right\}$.

(8) $\mathbb{A}^+ := \mathbb{A}^* \smallsetminus \{\boldsymbol{\varepsilon}\}$ is set of all non-empty finite words.

(9) $\mathbb{A}^\infty := \mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ is set of all finite and infinite words.

On words, we define the concatenation, in a natural way.

Definition 2.2. Let $\boldsymbol{u} \in \mathbb{A}^*$ be a finite word, and $\boldsymbol{v} \in \mathbb{A}^\infty$ a finite or infinite word. The **concatenation** of words $\boldsymbol{u}$ and $\boldsymbol{v}$ (denoted $\boldsymbol{u} \cdot \boldsymbol{v}$ or $\boldsymbol{uv}$) is a word of length $|\boldsymbol{u}| + |\boldsymbol{v}|$ defined as follows:

$$(\boldsymbol{u} \cdot \boldsymbol{v})_i := \begin{cases} u_i & \text{if} \quad 0 \le i < |\boldsymbol{u}| \\ v_{i-|\boldsymbol{u}|} & \text{if} \quad |\boldsymbol{u}| \le i < |\boldsymbol{u}| + |\boldsymbol{v}| \end{cases}$$

For any sets of words $\boldsymbol{X} \subseteq \mathbb{A}^*$ and $\boldsymbol{Y} \subseteq \mathbb{A}^\infty$ we write

$$\boldsymbol{X} \cdot \boldsymbol{Y} := \{\boldsymbol{xy} | \boldsymbol{x} \in \boldsymbol{X}, \boldsymbol{y} \in \boldsymbol{Y}\}.$$

Example. Words on the alphabet $\{0,1\}$ are for instance 01001 or 010. Holds that $01001 \cdot 010 = 01001010$.

Definition 2.3. Let $\boldsymbol{u} \in \mathbb{A}^*$, $k \in \mathbb{N}$. We define the **k-th power of word $\boldsymbol{u}$** recurrently:

(1) $\boldsymbol{u}^0 := \boldsymbol{\varepsilon}$;

(2) $\boldsymbol{u}^k := \boldsymbol{u}\boldsymbol{u}^{k-1}$ for any $k \geq 1$.

For $\boldsymbol{u} \neq \boldsymbol{\varepsilon}$ we define **infinite power** $\boldsymbol{u}^\omega \in \mathbb{A}^\mathbb{N}$ as $(\boldsymbol{u}^\omega)_i := u_{i-|\boldsymbol{u}|\lfloor i/|\boldsymbol{u}|\rfloor}$

Definition 2.4. Let $\boldsymbol{u} \in \mathbb{A}^\infty$, $\boldsymbol{v} \in \mathbb{A}^*$. We say that $\boldsymbol{v}$ is **factor** of $\boldsymbol{u}$, if there exist $\boldsymbol{s} \in \mathbb{A}^*$ and $\boldsymbol{t} \in \mathbb{A}^\infty$ such that $\boldsymbol{u} = \boldsymbol{svt}$.

Set of all factors of $\boldsymbol{u}$ is denoted by $\mathrm{Fact}(\boldsymbol{u})$.

Additionally, if $\boldsymbol{s} = \boldsymbol{\varepsilon}$ then $\boldsymbol{v}$ is called **prefix** of $\boldsymbol{u}$; and if $\boldsymbol{t} = \boldsymbol{\varepsilon}$ then $\boldsymbol{v}$ is called **suffix** of $\boldsymbol{u}$.

Let $n \in \mathbb{N}$. Set of all factors of word $\boldsymbol{u}$ of length n is denoted by $\mathrm{Fact}_n(\boldsymbol{u})$.

Example. $\mathrm{Fact}(01001) = \{\boldsymbol{\varepsilon}, 0, 1, 00, 01, 10, 001, 010, 100, 0100, 1001\}$.

Definition 2.5. Let $\boldsymbol{u}$ be a word on any given alphabet. For every a in the alphabet we define **number of occurrences** of letter a in $\boldsymbol{u}$ as

$$|\boldsymbol{u}|_a := \#\{i \in \widehat{|\boldsymbol{u}|} | u_i = a\}.$$

Example. $|01001|_0 = 3$ and $|01001|_1 = 2$.

For purpose of defining morphisms on infinite words, we need a metric on the set $\mathbb{A}^\infty$ of finite and infinite words. To do so, we will formally represent, by adding a new symbol $\bullet$ to the alphabet, a finite word $\boldsymbol{w} \in \mathbb{A}^*$ as an infinite word $\boldsymbol{w}\bullet^\omega$. Using this convention, we define:

Definition 2.6. For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{A}^\infty$, put

$$\mu(\boldsymbol{u}, \boldsymbol{v}) := \min\{i \in \mathbb{N} | \boldsymbol{u}_i \neq \boldsymbol{v}_i\}$$

and

$$d(\boldsymbol{u}, \boldsymbol{v}) := \frac{1}{1 + \mu(\boldsymbol{u}, \boldsymbol{v})}. \tag{2.1}$$

The mapping $d : \mathbb{A}^\infty \times \mathbb{A}^\infty \to [0,1]$ is called **metric** on $\mathbb{A}^\infty$.

Remark. **(1)** The value $\mu(\boldsymbol{u}, \boldsymbol{v})$ is the first index, where $\boldsymbol{u}$ differs from $\boldsymbol{v}$.

(2) If $\boldsymbol{u} = \boldsymbol{v}$, then the set $\{i \in \mathbb{N} | \boldsymbol{u}_i \neq \boldsymbol{v}_i\}$ is empty and we have $\mu(\boldsymbol{u}, \boldsymbol{u}) = \min \emptyset = +\infty$.

The following proposition shows that the name "metric" for mapping d is justified.

Lemma 2.7. *Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{A}^\infty$. Then*

$$\mu(\boldsymbol{u}, \boldsymbol{w}) \geq \min\{\mu(\boldsymbol{u}, \boldsymbol{v}), \mu(\boldsymbol{v}, \boldsymbol{w})\}.$$

Proof. Let $k := \mu(\boldsymbol{u}, \boldsymbol{w})$. Then $u_k \neq w_k$, and thus either $v_k \neq u_k$ or $v_k \neq w_k$. Hence $\min\{\mu(\boldsymbol{u}, \boldsymbol{v}), \mu(\boldsymbol{v}, \boldsymbol{w})\} \leq k$. □

Proposition 2.8. *The mapping d defined by* (2.1) *is a metric on $\mathbb{A}^\infty$.*

Proof. **(1)** For $\boldsymbol{u} \neq \boldsymbol{v}$ we have $\mu(\boldsymbol{u}, \boldsymbol{v}) < +\infty$, hence $d(\boldsymbol{u}, \boldsymbol{v}) > 0$. The equality $d(\boldsymbol{u}, \boldsymbol{u}) = 0$ applies by definition.

(2) The function μ is symmetric hence the function d is symmetric.

(3) The function d is decreasing function of μ, which with previous lemma gives $d(\boldsymbol{u}, \boldsymbol{w}) \leq \max\{d(\boldsymbol{u}, \boldsymbol{v}), d(\boldsymbol{v}, \boldsymbol{w})\} \leq d(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, \boldsymbol{w})$. □

Property 2.9. *$\mathbb{A}^\mathbb{N}$ with the metric d is a compact space.*

Proof. For contradiction, suppose that $\mathbb{A}^\mathbb{N}$ with d is not compact. That means that there exists an open cover $\mathcal{S}$ of $\mathbb{A}^\mathbb{N}$ which does not have finite subcover. Hence as $\left\{a\mathbb{A}^\mathbb{N} \middle| a \in \mathbb{A}\right\}$ is a partition[1] of $\mathbb{A}^\mathbb{N}$, one of its members does not have finite subcover. Let us denote this one $M_0 = w_0\mathbb{A}^\mathbb{N}$. Surely $d(M_0) = \frac{1}{2}$.

We know that $\left\{w_0 a\mathbb{A}^\mathbb{N} \middle| a \in \mathbb{A}\right\}$ is a partition of M_0, hence exists letter w_1 such that $M_1 := w_0 w_1 \mathbb{A}^\mathbb{N}$ does not have finite subcover and holds that $d(M_1) = \frac{1}{3}$. This way we can construct infinite word $\boldsymbol{w} = w_0 w_1 w_2 \ldots$ which belongs to M_n for every $n \in \mathbb{N}$. Denote $N \in \mathcal{S}$ set in the primary cover $\mathcal{S}$ which owns $\boldsymbol{w}$. As N is open, it owns some open ball $B(\boldsymbol{w}, r)$ and $M_n \subseteq B(\boldsymbol{w}, r)$ for large enough n. This is in contradiction with the fact that M_n does not have finite subcover of $\mathcal{S}$. □

Property 2.10. *The set $\mathbb{A}^*$ is an associative monoid with cancellation, i.e.:*

(1) $(\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{A}^*)(\boldsymbol{uv} \in \mathbb{A}^*)$;

[1] i.e. $\mathbb{A}^\mathbb{N} = \bigcup\limits_{a \in \mathbb{A}} a\mathbb{A}^\mathbb{N}$

(2) *the empty word* $\boldsymbol{\varepsilon}$ *is a neutral element of* $\mathbb{A}^*$*:* $(\forall \boldsymbol{u} \in \mathbb{A}^*)(\boldsymbol{u} = \boldsymbol{u}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\boldsymbol{u})$*;*

(3) $(\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{A}^*)\big((\boldsymbol{u}\boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v}\boldsymbol{w})\big)$*;*

(4) $(\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{s} \in \mathbb{A}^*)\big((\boldsymbol{u}\boldsymbol{s} = \boldsymbol{v}\boldsymbol{s} \vee \boldsymbol{s}\boldsymbol{u} = \boldsymbol{s}\boldsymbol{v}) \implies \boldsymbol{u} = \boldsymbol{v}\big)$.

Proof. Claims 1–3 are proven from definition of concatenation.

Let us prove 4, by contradiction, and let us do the proof just for the case $\boldsymbol{u}\boldsymbol{s} = \boldsymbol{v}\boldsymbol{s}$, as the second case is very similar. First, if $|\boldsymbol{u}| \neq |\boldsymbol{v}|$, we get $|\boldsymbol{u}\boldsymbol{s}| \neq |\boldsymbol{v}\boldsymbol{s}|$, which is in contradiction with equality of $\boldsymbol{u}\boldsymbol{s}$ and $\boldsymbol{v}\boldsymbol{s}$ as maps (they have different length, i.e. different domain). So $|\boldsymbol{u}| = |\boldsymbol{v}|$, but $\big(\exists i \in \widehat{|\boldsymbol{u}|}\big)(u_i \neq v_i)$. Hence $(\boldsymbol{u}\boldsymbol{s})_i \neq (\boldsymbol{v}\boldsymbol{s})_i$, contradiction. □

2.2 Morphisms on words

Definition 2.11. Let φ be a mapping $\varphi : \mathbb{A}^\infty \to \mathbb{A}^\infty$. We say that φ is a **morphism** over alphabet $\mathbb{A}$ when φ satisfies:

(1) $(\forall a \in \mathbb{A})(\varphi(a) \neq \boldsymbol{\varepsilon})$;

(2) $(\forall \boldsymbol{u} \in \mathbb{A}^*)(\forall \boldsymbol{v} \in \mathbb{A}^\infty)(\varphi(\boldsymbol{u}\boldsymbol{v}) = \varphi(\boldsymbol{u})\varphi(\boldsymbol{v}))$.

Remark. **(1)** The set $\mathbb{A}^\infty$ is not a monoid, hence φ is not a morphism in algebra. However, as $\mathbb{A}^*$ is a monoid, φ restricted to $\mathbb{A}^*$ is a morphism $\varphi : \mathbb{A}^* \to \mathbb{A}^*$. And as will be shown below, $\varphi : \mathbb{A}^\infty \to \mathbb{A}^\infty$ is a continuous expansion of $\varphi : \mathbb{A}^* \to \mathbb{A}^*$.

(2) In the language of algebra, the mapping φ would be a morphism, even if $\varphi(a) = \boldsymbol{\varepsilon}$ for some letter $a \in \mathbb{A}$. Some literature call morphisms in our definition as non-erasing. But as we do not work with erasing morphisms, we will not specify this.

Property 2.12. *Any morphism over* $\mathbb{A}$ *is continuous.*

Proof. We will show that $d(\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})) \leq d(\boldsymbol{u}, \boldsymbol{v})$. If $\boldsymbol{u} = \boldsymbol{v}$, the claim holds. Let $\boldsymbol{u} \neq \boldsymbol{v}$. First, from definition of morphism we have that $|\varphi(a)| \geq 1$. Mathematical induction by the length $|\boldsymbol{x}|$ gives $|\varphi(\boldsymbol{x})| \geq |\boldsymbol{x}|$. Now let $\mu(\boldsymbol{u}, \boldsymbol{v}) = k$, hence exists $\boldsymbol{s} \in \mathbb{A}^k$ such that $\boldsymbol{u} = \boldsymbol{s}\boldsymbol{u'}$ and $\boldsymbol{v} = \boldsymbol{s}\boldsymbol{v'}$. Then $\mu(\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})) \geq |\varphi(\boldsymbol{s})| \geq |\boldsymbol{s}| = \mu(\boldsymbol{u}, \boldsymbol{v})$ and as d is descending function of μ, holds that $d(\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})) \leq d(\boldsymbol{u}, \boldsymbol{v})$. □

Property 2.13. *The morphism is well-defined by its value on single-letter words, i.e.*

$$(\forall \varphi, \psi \text{ morphisms over } \mathbb{A})\Big(\big(\forall a \in \mathbb{A}\big)\big(\varphi(a) = \psi(a)\big) \implies \varphi = \psi\Big).$$

PROOF. We need to show that if condition is true for two morphisms φ, ψ, then for every word $\boldsymbol{u} \in \mathbb{A}^\infty$ holds $\varphi(\boldsymbol{u}) = \psi(\boldsymbol{u})$.

For finite $\boldsymbol{u}$ we show the claim by mathematical induction on $n = |\boldsymbol{u}|$.

($n = 0$) For every morphism holds that $\varphi(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}$.

($n \to n+1$) Put $\boldsymbol{u} = \boldsymbol{w}a$, $a \in \mathbb{A}$. Then

$$\varphi(\boldsymbol{u}) = \varphi(\boldsymbol{w}a) = \varphi(\boldsymbol{w})\varphi(a) = \psi(\boldsymbol{w})\psi(a) = \psi(\boldsymbol{u}).$$

Let $\boldsymbol{w}$ be an infinite word and let $\boldsymbol{u}^{(n)}$ be the prefix of the word $\boldsymbol{w}$ of the length n. Then $\lim_{n\to\infty} \boldsymbol{u}^{(n)} = \boldsymbol{w}$ (limit in the sense of metric d). As $\varphi(\boldsymbol{u}^{(n)}) = \psi(\boldsymbol{u}^{(n)})$ for all $n \in \mathbb{N}$ and the morphisms are continuous, we have $\varphi(\boldsymbol{w}) = \psi(\boldsymbol{w})$. □

According to this property, we can use the following convention.

Convention. The following notation is used to define morphism φ over the alphabet $\mathbb{A} = \{a_0, \dots, a_{k-1}\}$:

$$\varphi : \begin{array}{rcl} a_0 & \mapsto & \varphi(a_0) \\ a_1 & \mapsto & \varphi(a_1) \\ \vdots & & \vdots \\ a_{k-1} & \mapsto & \varphi(a_{k-1}) \end{array}$$

Definition 2.14. We define following important morphisms over alphabet $\{0, 1\}$:

$$I : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 1 \end{array}; \qquad E : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}; \qquad F : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}; \qquad \widetilde{F} : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}. \tag{2.2}$$

Theorem 2.15. *Let φ be a morphism over $\mathbb{A}$. Then φ has a fixed point starting with $a \in \mathbb{A}$, if and only if, $\varphi(a)$ starts with letter a.*

Moreover, if $|\varphi(a)| \geq 2$, the fixed point is unique (for given φ and a).

PROOF. Let $\varphi(a) = a\boldsymbol{x}$. The following two cases are discussed.

($\boldsymbol{x} = \boldsymbol{\varepsilon}$) Then $\varphi(a) = a$ and $\varphi(a^\omega) = a^\omega$. Hence $\boldsymbol{u} = a^\omega$ is a fixed point of φ.

($\boldsymbol{x} \neq \boldsymbol{\varepsilon}$) Put $\boldsymbol{u}^{(0)} := a$ as an one-letter word. Define sequence $\boldsymbol{u}^{(n)}$ recurrently putting

$$\boldsymbol{u}^{(n+1)} := \varphi\big(\boldsymbol{u}^{(n)}\big). \tag{2.3}$$

k	$\boldsymbol{f}^{(k)} := F(\boldsymbol{f}^{(k-1)})$
0	1
1	0
2	01
3	010
4	01001
5	01001010
6	0100101001001
7	010010100100101001010
$\vdots$	$\ddots$

Table 2.1: Iteration of Fibonacci morphism F.

First, we show that $\boldsymbol{u}^{(n)}$ is a prefix of $\boldsymbol{u}^{(n+1)}$, by mathematical induction. Equalities $\boldsymbol{u}^{(0)} = a$ and $\boldsymbol{u}^{(1)} = \varphi(a) = a\boldsymbol{x}$ give us the claim for $n = 0$. Now let $\boldsymbol{u}^{(n)} = \boldsymbol{u}^{(n-1)}\boldsymbol{y}$. Then $\boldsymbol{u}^{(n+1)} = \varphi\big(\boldsymbol{u}^{(n)}\big) = \varphi\big(\boldsymbol{u}^{(n-1)}\boldsymbol{y}\big) = \boldsymbol{u}^{(n)}\varphi(\boldsymbol{y})$, giving the claim.

One-letter word a is a factor of $\boldsymbol{u}^{(n)}$ for every n, and $|\varphi(a)| > |a|$, hence $\big|\boldsymbol{u}^{(n+1)}\big| > \big|\boldsymbol{u}^{(n)}\big|$. Along with the previous observation about prefixes we get that sequence $\boldsymbol{u}^{(n)}$ is convergent, and we can put $\boldsymbol{u} = \lim_{n\to+\infty} \boldsymbol{u}^{(n)}$. Applying limit on the equation (2.3) gives that $\varphi(\boldsymbol{u}) = \boldsymbol{u}$, proving the first part of theorem.

Moreover, it is clear that a fixed point must start with such letter a that $\varphi(a)$ starts by a. We have shown that for every such a there exists a fixed point.

Finally we show the uniqueness of the fixed point for every a in the case that $|\varphi(a)| \geq 2$. Let $a\boldsymbol{w}$ be an infinite word such that $\varphi(a\boldsymbol{w}) = a\boldsymbol{w}$. Then $a\boldsymbol{w} = \varphi^k(a\boldsymbol{w}) = \boldsymbol{u}^{(k)}\varphi^k(\boldsymbol{w})$, which means that $\boldsymbol{u}^{(k)}$ is a prefix of $\boldsymbol{w}$ for every k, hence $\boldsymbol{w} = \lim_{k\to+\infty} \boldsymbol{u}^{(k)} = \boldsymbol{u}$. $\square$

Example. Let us take the morphism F defined by (2.2). We have that $F(0)$ is a 2-letter word and starts with 0. Moreover, $F(1)$ does not start with 1, which means, that F has exactly one fixed point $\boldsymbol{f}$, so-called **Fibonacci word**. Its construction is seen in Table 2.1. The beginning of this infinite word looks as follows:

$$\boldsymbol{f} = 0100101001001010010100100101001001\cdots \tag{2.4}$$

For the examination of sturmian morphisms, we will widely use their matrices, whose determinant is equal to 1 or -1. We now define a matrix of morphism and show the most important properties.

Definition 2.16. Let $\mathbb{A}$ be an alphabet, let us sort the letters in alphabet $\mathbb{A}$, put $\mathbb{A} = \{a_0, a_1, \dots, a_{k-1}\}$, where $k = \#\,\mathbb{A}$ is the size of the alphabet.

(1) Let $n \in \mathbb{N}$, $n \geq 1$, and let $\boldsymbol{u}^{(0)}, \dots, \boldsymbol{u}^{(n-1)} \in \mathbb{A}^*$ be n finite words over $\mathbb{A}$. We define **matrix of words** $\boldsymbol{u}^{(0)}, \dots \boldsymbol{u}^{(n-1)}$ of order $k \times n$ as

$$\mathbf{M}\left(\boldsymbol{u}^{(0)}, \dots, \boldsymbol{u}^{(n-1)}\right) := \begin{pmatrix} \left|\boldsymbol{u}^{(0)}\right|_{a_0} & \left|\boldsymbol{u}^{(1)}\right|_{a_0} & \dots & \left|\boldsymbol{u}^{(n-1)}\right|_{a_0} \\ \left|\boldsymbol{u}^{(0)}\right|_{a_1} & \left|\boldsymbol{u}^{(1)}\right|_{a_1} & \dots & \left|\boldsymbol{u}^{(n-1)}\right|_{a_1} \\ \dots & \dots & \dots & \dots \\ \left|\boldsymbol{u}^{(0)}\right|_{a_{k-1}} & \left|\boldsymbol{u}^{(1)}\right|_{a_{k-1}} & \dots & \left|\boldsymbol{u}^{(n-1)}\right|_{a_{k-1}} \end{pmatrix}.$$

Note that for $n = 1$, $\mathbf{M}\left(\boldsymbol{u}^{(0)}\right)$ is a column.

(2) Let φ be a morphism over $\mathbb{A}$. We define **matrix of morphism** φ as

$$\begin{aligned} \mathbf{M}_\varphi &:= \mathbf{M}\left(\varphi(a_0), \varphi(a_1), \dots, \varphi(a_{k-1})\right) \\ &= \begin{pmatrix} |\varphi(a_0)|_{a_0} & |\varphi(a_1)|_{a_0} & \dots & |\varphi(a_{k-1})|_{a_0} \\ |\varphi(a_0)|_{a_1} & |\varphi(a_1)|_{a_1} & & |\varphi(a_{k-1})|_{a_1} \\ \vdots & & \ddots & \vdots \\ |\varphi(a_0)|_{a_{k-1}} & |\varphi(a_1)|_{a_{k-1}} & \dots & |\varphi(a_{k-1})|_{a_{k-1}} \end{pmatrix}. \end{aligned}$$

Example. Let us write the matrix of words and morphisms for alphabet $\mathbb{A} = \{0, 1\}$:

$$\begin{aligned} \mathbf{M}\left(\boldsymbol{u}^{(0)}, \dots, \boldsymbol{u}^{(n-1)}\right) &= \begin{pmatrix} \left|\boldsymbol{u}^{(0)}\right|_0 & \dots & \left|\boldsymbol{u}^{(n-1)}\right|_0 \\ \left|\boldsymbol{u}^{(0)}\right|_1 & \dots & \left|\boldsymbol{u}^{(n-1)}\right|_1 \end{pmatrix}; \\ \mathbf{M}_\varphi = \mathbf{M}\left(\varphi(0), \varphi(1)\right) &= \begin{pmatrix} |\varphi(0)|_0 & |\varphi(1)|_0 \\ |\varphi(0)|_1 & |\varphi(1)|_1 \end{pmatrix}. \end{aligned}$$

Example. Matrices of morphisms defined by (2.2) are

$$\mathbf{M}_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \mathbf{M}_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \mathbf{M}_F = \mathbf{M}_{\widetilde{F}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 2.17. *Let $\boldsymbol{u} \in \mathbb{A}^*$ be a finite word and let φ, ψ be morphisms over $\mathbb{A} = \{a_0, a_1, \dots, a_{k-1}\}$. Then*

(1) $\mathbf{M}\left(\varphi(\boldsymbol{u})\right) = \mathbf{M}_\varphi \cdot \mathbf{M}\left(\boldsymbol{u}\right)$;

(2) $\mathbf{M}_{\varphi\psi} = \mathbf{M}_\varphi \mathbf{M}_\psi$.

Proof. **(1)** For all $i \in \widehat{n}$ holds that

$$\begin{aligned}\mathbf{M}\left(\varphi(\boldsymbol{u})\right)_{i0} = \left|\varphi(\boldsymbol{u})\right|_{a_i} = \sum_{j=0}^{k-1} \left|\varphi(a_j)\right|_{a_i} \left|\boldsymbol{u}\right|_{a_j} = \sum_{j=0}^{k-1} \left(\mathbf{M}_\varphi\right)_{ij} \mathbf{M}\left(\boldsymbol{u}\right)_{j0} \\ = \left(\mathbf{M}_\varphi \cdot \mathbf{M}\left(\boldsymbol{u}\right)\right)_{i0}.\end{aligned}$$

(2) For all $i, j \in \widehat{n}$ holds that

$$\begin{aligned}\left(\mathbf{M}_{\varphi\psi}\right)_{ij} = \left|\varphi\psi(a_j)\right|_{a_i} = \mathbf{M}\left(\varphi(\psi(a_j))\right)_{i0} = \left(\mathbf{M}_\varphi \cdot \mathbf{M}\left(\psi(a_j)\right)\right)_{i0} \\ = \Big(\mathbf{M}_\varphi \mathbf{M}_\psi \cdot \underbrace{\mathbf{M}\left(a_j\right)}_{\mathbf{c}^{(j)}}\Big)_{i0} = \left(\mathbf{M}_\varphi \mathbf{M}_\psi\right)_{ij},\end{aligned}$$

where $\mathbf{c}^{(j)}$ is a vector of the standard basis, $\left(\mathbf{c}^{(j)}\right)_i = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$.

$\square$

Chapter 3

Sturmian Words and Morphisms

Sturmian words can be defined in different equivalent ways. As we study two- and three-interval exchange transformations, we will prefer the definition by 2iet words.

3.1 Interval exchange transformation

Interval exchange transformation can be defined very generally for any number of intervals and any permutation of the intervals. For purposes of this thesis, just 2- and 3-interval exchange is needed.

Definition 3.1. Let $\alpha \in (0,1)$. We define **2-interval exchange transformation** $T_\alpha : [0,1) \to [0,1)$ by

$$T_\alpha(\xi) := \begin{cases} \xi + 1 - \alpha & \text{if} \quad \xi < \alpha \\ \xi - \alpha & \text{if} \quad \xi \geq \alpha \end{cases}$$

(the action of the mapping T_α is shown in Figures 3.1 and 3.2). Number α is called the **parameter** (or **slope**) of the transformation T_α.

Definition 3.2. Let T_α be a 2-interval exchange transformation, let $\rho \in [0,1)$, and let $\ell \in \mathbb{N} \cup \{\infty\}$. We define a word $\boldsymbol{t}$ of length ℓ on the alphabet $\{0,1\}$ as follows:

$$t_i := \begin{cases} 0 & \text{if} \quad T_\alpha^i(\rho) < \alpha \\ 1 & \text{if} \quad T_\alpha^i(\rho) \geq \alpha \end{cases},$$

and we write

$$\boldsymbol{t} \sim (T_\alpha^i(\rho))_{i=0}^{\ell}. \tag{3.1}$$

Infinite word $\boldsymbol{t}$ is called a **2iet word**, finite word $\boldsymbol{t}$ is called a **2iet factor**.

The number ρ is called the **start point** (or **intercept**) of the 2iet word or factor.

The following proposition follows directly from the definition.

Proposition 3.3. *Let T_α be a 2-interval exchange transformation, let $\rho \in [0,1)$. Then*

$$T_\alpha^k(\rho) \equiv \rho - k\alpha \pmod 1.$$

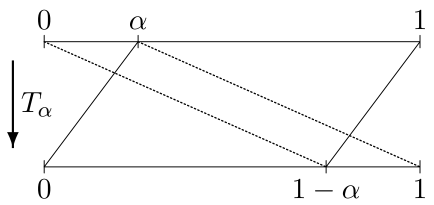

Figure 3.1: Two-interval exchange transformation T_α

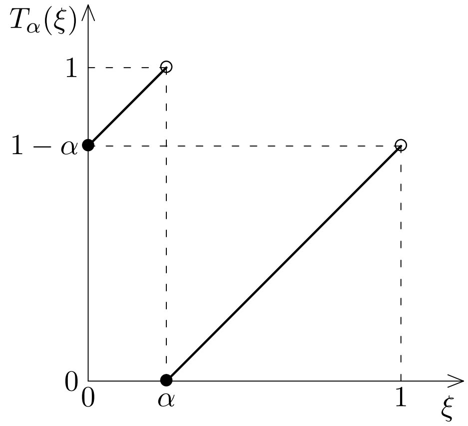

Figure 3.2: Graph of two-interval exchange transformation T_α

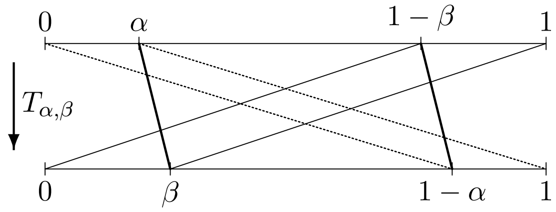

Figure 3.3: Three-interval exchange transformation $T_{\alpha,\beta}$

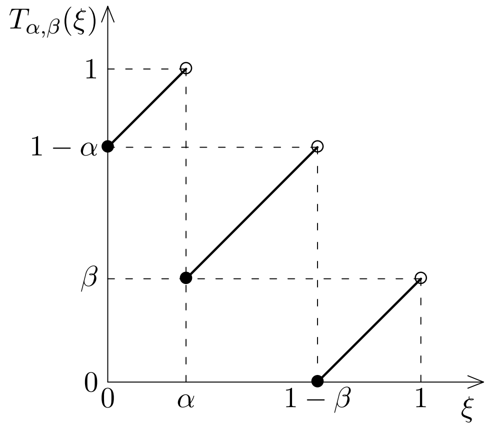

Figure 3.4: Graph of three-interval exchange transformation $T_{\alpha,\beta}$

k	$T_\tau^k(1-\tau)$	$(\boldsymbol{t}_{\tau,1-\tau})_k$	k	$T_\tau^k(1-\tau)$	$(\boldsymbol{t}_{\tau,1-\tau})_k$
0	0.382	0	12	0.966	1
1	0.764	1	13	0.348	0
2	0.146	0	14	0.729	1
3	0.528	0	15	0.111	0
4	0.910	1	16	0.493	0
5	0.292	0	17	0.875	1
6	0.674	1	18	0.257	0
7	0.056	0	19	0.639	1
8	0.438	0	20	0.021	0
9	0.820	1	21	0.403	0
10	0.202	0	22	0.785	1
11	0.584	0		$\vdots$	

Table 3.1: Example of 2iet word $\boldsymbol{t} \sim (T_\tau^i(1-\tau))_{i=0}^{\infty}$.

Example. In Table 3.1, there is an example of 2-interval exchange transformation and 2iet word for parameter $\alpha = \tau := \frac{\sqrt{5}-1}{2} \approx 0.618$ and start point $\rho = 1-\tau \approx 0.382$. Number τ is often called the **golden ratio** and $\boldsymbol{t} \sim (T_\tau^i(1-\tau))_{i=0}^{\infty}$ is the Fibonacci word, see example after Theorem 2.15.

Definition 3.4. Let $\alpha, \beta \in (0,1)$ be real numbers such that $\alpha + \beta < 1$. We define **3-interval exchange transformation** $T_{\alpha,\beta} : [0,1) \to [0,1)$ as

$$T_{\alpha,\beta}(\xi) := \begin{cases} \xi + 1 - \alpha & \text{if} \quad \xi \in [0,\alpha) \\ \xi + \beta - \alpha & \text{if} \quad \xi \in [\alpha, 1-\beta) \\ \xi + \beta - 1 & \text{if} \quad \xi \in [1-\beta, 1) \end{cases}$$

(the action of the mapping $T_{\alpha,\beta}$ is shown in Figures 3.3 and 3.4).

Definition 3.5. Let $T_{\alpha,\beta}$ be a 3-interval exchange transformation, and let $\rho \in [0,1)$. To transformation $T_{\alpha,\beta}$ and number ρ, we assign an infinite word $\boldsymbol{t}$ as follows:

$$t_i := \begin{cases} 0 & \text{if} \quad T_{\alpha,\beta}^i(\rho) \in [0,\alpha) \\ 2 & \text{if} \quad T_{\alpha,\beta}^i(\rho) \in [\alpha, 1-\beta) \\ 1 & \text{if} \quad T_{\alpha,\beta}^i(\rho) \in [1-\beta, 1) \end{cases},$$

and we write

$$\boldsymbol{t} \sim (T_{\alpha,\beta}^i(\rho))_{i=0}^{\infty}. \tag{3.2}$$

If the word $\boldsymbol{t}$ is aperiodic, we call it a **3iet word**. Moreover, we call it **degenerate**, if $1 \in (1-\alpha)\mathbb{Z} + (1-\beta)\mathbb{Z}$, and **non-degenerate** otherwise.

Remark. We ought to note that the definitions of 2iet and 3iet words are a bit inconsistent, because we do allow periodic 2iet words, but we do not allow periodic 3iet words. Reason for this is that aperiodic 2iet words are in fact sturmian words, and we widely use periodic 2iet words in the following text.

3.2 Language of two-interval exchange transformation

Definition 3.6. Let $N \in \mathbb{N}$ be a natural number and let $\alpha \in [0,1)$. We define the set of words $\mathcal{L}_N(\alpha)$ as follows:

$$\mathcal{L}_N(\alpha) := \mathrm{Fact}_N(\boldsymbol{t}),$$

where $\boldsymbol{t} \sim (T_\alpha^i(0))_{i=0}^{\infty}$ is a 2iet word with parameter α.

The union of sets $\mathcal{L}_N(\alpha)$ for all $N \in \mathbb{N}$ is called a **language** of transformation T_α:

$$\mathcal{L}(\alpha) := \bigcup_{N \in \mathbb{N}} \mathcal{L}_N(\alpha) = \mathrm{Fact}(\boldsymbol{t}).$$ [1]

Definition 3.7. Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^{\infty}$ be two words of the same length. We say that $\boldsymbol{u}$ is **lexicographically smaller** than $\boldsymbol{v}$ (denoted $\boldsymbol{u} \prec \boldsymbol{v}$), if there exist words $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{u} = \boldsymbol{w}0\boldsymbol{x}$ and $\boldsymbol{v} = \boldsymbol{w}1\boldsymbol{y}$. Similarly we define relations $\succ$, $\preceq$ and $\succeq$.

The following properties follow easily from the definition.

Property 3.8. *Lexicographical order is an order, i.e.:*

(1) $(\forall \boldsymbol{u} \in \{0,1\}^{\infty})(\boldsymbol{u} \preceq \boldsymbol{u})$ *(reflexivity);*

(2) $(\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \{0,1\}^{\infty})((\boldsymbol{u} \preceq \boldsymbol{v} \wedge \boldsymbol{v} \preceq \boldsymbol{w}) \implies \boldsymbol{u} \preceq \boldsymbol{w})$ *(transitivity);*

(3) $(\forall \boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^{\infty}, \boldsymbol{u} \neq \boldsymbol{v})(\boldsymbol{u} \preceq \boldsymbol{v} \implies \boldsymbol{v} \not\preceq \boldsymbol{u})$ *(anti-symmetry).*

The following proposition is very important for counting pairs of amicable sturmian morphisms.

Proposition 3.9. *Let $p, N \in \mathbb{N}$ be coprime natural numbers such that $p \leq N$. For every $\rho \in [0,1)$ put $\boldsymbol{t}_\rho$ the 2iet factor of length N given by*

$$\boldsymbol{t}_\rho \sim (T_{p/N}^i(\rho))_{i=0}^{N-1}.$$

Then the following holds.

[1] It can be easily shown [3] that the the language does not depend on the start point ρ, but depends only on the parameter α, i.e. $\mathrm{Fact}(\boldsymbol{t}_{\alpha,\rho}) = \mathrm{Fact}(\boldsymbol{t}_{\alpha,\rho'})$.

(1) *Suppose $j \in \widehat{N}$. Then for any $\rho, \rho' \in \left[\frac{j}{N}, \frac{j+1}{N}\right)$ holds that $\boldsymbol{t}_\rho = \boldsymbol{t}_{\rho'}$.*

(2) *Let $j \in \widehat{N}$. Then the mapping $f : \widehat{N} \to \{0, 1/N, \dots, (N-1)/N\}$ given by*

$$f(k) := T^k_{p/N}(j/N)$$

is injective.

(3) *Suppose $j, j' \in \mathbb{N}$ such that $0 \le j' < j < N$. Then*

$$\boldsymbol{t}_{j'/N} \prec \boldsymbol{t}_{j/N}.$$

(4) *The number of elements of the set $\mathcal{L}_N(p/N)$ is*

$$\# \mathcal{L}_N(p/N) = N.$$

Proof. Let us shorten $T := T_{p/N}$.

(1) We need to show that $T^k(\rho) < p/N$, if and only if, $T^k(j/N) < p/N$. But $T^k(j/N) = i/N$ for some $i \in \widehat{N}$ and $T^k(\rho) = i/N + (\rho - j/N)$, because $\rho - j/N < 1/N$ and $i/N < (N-1)/N$. That gives $\boldsymbol{t}_\rho = \boldsymbol{t}_{j/N}$ for all $\rho \in \left[\frac{j}{N}, \frac{j+1}{N}\right)$

This means that we can represent all possible words in $\mathcal{L}_N(p/N)$ by words with $\rho \in \{0, 1/N, \dots, (N-1)/N\}$.

(2) We will find the inverse mapping to f. Proposition 3.3 gives in this case

$$N \cdot T^k(j/N) \equiv j - pk \pmod{N}.$$

As p, N are coprime, there exists $m \in \widehat{N}$ such that

$$pm \equiv 1 \pmod{N}.$$

From these two congruences, we get

$$k \equiv m\big(j - N \cdot T^k(j/N)\big) \pmod{N}$$

This means that we found the inverse mapping

$$f^{-1}(i/N) \equiv m(j - i) \pmod{N},$$

which proves the injectiveness.

(3) Due to transitivity of relation $(\prec)$, it is enough to show the claim for $j' = j - 1$. According to the previous, there exist exactly one k_0 and k_1 such that

$$T^{k_0}(j/N) = 0 \qquad \text{and} \qquad T^{k_1}(j/N) = p/N,$$

which means that

$$T^{k_0}(j'/N) = (N-1)/N \qquad \text{and} \qquad T^{k_1}(j'/N) = (p-1)/N.$$

We know that at all positions but k_0 and k_1, the words have same letters. But we can see that either $k_0 = k_1 + 1$, which means that $\boldsymbol{t}_{j'/N} = \boldsymbol{x}01\boldsymbol{y}$ and $\boldsymbol{t}_{j/N} = \boldsymbol{x}10\boldsymbol{y}$, or $k_0 = 0$ and $k_1 = N - 1$, but this would mean that $j = 0$ and $j' = N - 1$, which we do not allow.

(4) We can see that $\mathcal{L}_N(p/N) = \left\{\boldsymbol{t}_\rho \middle| \rho \in [0,1)\right\}$. The interval $[0,1)$ is divided into N sub-intervals, we know that each of them is assigned one word in $\mathcal{L}_N(p/N)$, and due to strict lexicographical order of these words, they are different for different intervals. □

3.3 Sturmian words and their equivalent definitions

There exist many equivalent definitions of sturmian words. We will define sturmian words using the balance property.

Definition 3.10. Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^*$ be words of the same finite length. We define $\delta(\boldsymbol{u}, \boldsymbol{v}) := \left||\boldsymbol{u}|_0 - |\boldsymbol{v}|_0\right|$

Definition 3.11. Let $\boldsymbol{U} \subseteq \{0,1\}^*$ be a set of words. We say that the set $\boldsymbol{U}$ is balanced, if

$$(\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{U}, |\boldsymbol{x}| = |\boldsymbol{y}|)(\delta(\boldsymbol{x}, \boldsymbol{y}) \leq 1).$$

Let $\boldsymbol{u} \in \{0,1\}^\infty$ be a finite or infinite word. We say that $\boldsymbol{u}$ is **balanced**, if the set $\mathrm{Fact}(\boldsymbol{u})$ is balanced.

Definition 3.12. Let $\boldsymbol{u} \in \mathbb{A}^\mathbb{N}$ be an infinite word.

(1) We say that $\boldsymbol{u}$ is **periodic**, if there exist finite words $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{A}^*$ such that $\boldsymbol{u} = \boldsymbol{v}\boldsymbol{w}^\omega$. Moreover, if $\boldsymbol{u} = \boldsymbol{w}^\omega$, we say that $\boldsymbol{u}$ is **strictly periodic**.

(2) We say that $\boldsymbol{u}$ is **aperiodic**, if $\boldsymbol{u}$ is not periodic.

Definition 3.13. Let $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ be an infinite word. We say that $\boldsymbol{u}$ is **sturmian**, if $\boldsymbol{u}$ is balanced and aperiodic.

Definition 3.14. We define a **complexity** of words on alphabet $\{0,1\}$ as

$$\mathrm{C}(\boldsymbol{u}, n) := \#\operatorname{Fact}_n(\boldsymbol{u})$$

for all $n \in \mathbb{N}$. That means that $\mathrm{C}(\boldsymbol{u}, n)$ is number of factors of $\boldsymbol{u}$ of length n.

The following theorem [3] allows us to describe the sturmian words by their complexity.

Theorem 3.15. *Let $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ be an infinite word. Then $\boldsymbol{u}$ is sturmian, if an only if,*

$$(\forall n \in \mathbb{N})(\mathrm{C}(\boldsymbol{u}, n) = n + 1).$$

Definition 3.16. Let $\boldsymbol{u} \in \{0,1\}^{\infty}$ be a finite or infinite word and let $\boldsymbol{x}$ be its factor. We say that $\boldsymbol{x}$ is **right special** factor of $\boldsymbol{u}$, if and only if, both $\boldsymbol{x}0$ and $\boldsymbol{x}1$ are factors of $\boldsymbol{u}$.

Lemma 3.17. *Let $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ be an infinite word, $n \in \mathbb{N}$. Then number of right special factors of length n is equal to $\mathrm{C}(\boldsymbol{u}, n+1) - \mathrm{C}(\boldsymbol{u}, n)$.*

Proof. Let R be set of factors of length n that are right special and S set of those that are not. For $\boldsymbol{x} \in R$ both $\boldsymbol{x}0$ and $\boldsymbol{x}1$ are factors of $\boldsymbol{u}$, for $\boldsymbol{x} \in S$ exactly one of $\boldsymbol{x}1$ and $\boldsymbol{x}0$ is factor of $\boldsymbol{u}$. Hence $\mathrm{C}(\boldsymbol{u}, n+1) - \mathrm{C}(\boldsymbol{u}, n) = (2 \cdot \# R + \# S) - (\# R + \# S) = \# R$. □

Proposition 3.18. *Let $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$. Then $\boldsymbol{u}$ is sturmian, if and only if, it contains exactly one right-special factor of each length.*

Proof. **($\Rightarrow$)** From Theorem 3.15 we have $\mathrm{C}(\boldsymbol{u}, n+1) - \mathrm{C}(\boldsymbol{u}, n) = (n+2) - (n+1) = 1$.

($\Leftarrow$) Equalities $\mathrm{C}(\boldsymbol{u}, 0) = 1$ and $\mathrm{C}(\boldsymbol{u}, n+1) - \mathrm{C}(\boldsymbol{u}, n) = 1$ follows to $\mathrm{C}(\boldsymbol{u}, n) = n + 1$ for every $n \in \mathbb{N}$. □

The most important for us is characterization of sturmian words by 2-interval exchange transformation.

Theorem 3.19. *Let $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ be an infinite word. Then $\boldsymbol{u}$ is sturmian, if and only if, there exists $\alpha, \rho \in [0,1)$ such that α is irrational and*

$$\boldsymbol{u} \sim (T_\alpha^i(\rho))_{i=0}^{\infty} \quad \textit{or} \quad E(\boldsymbol{u}) \sim (T_\alpha^i(\rho))_{i=0}^{\infty}, \tag{3.3}$$

where E is morphism $E : \begin{array}{c} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$.

Remark. If the first holds in (3.3), then the word $\boldsymbol{u}$ is sometimes called **lower mechanical**. If the second holds, it is called **upper mechanical**.

Proposition 3.20. *Let $\alpha \in [0,1)$. Then the set $\mathcal{L}(\alpha)$ is balanced.*

Moreover, suppose α rational, $\alpha = p/N$. Then for each $\boldsymbol{v} \in \mathcal{L}_N(p/N)$ the periodic infinite word $\boldsymbol{v}^\omega$ is balanced.

Proof. If α is irrational, then $\mathcal{L}(\alpha) = \mathrm{Fact}(\boldsymbol{u})$ for some sturmian word $\boldsymbol{u}$, and hence the set is balanced.

To prove the second claim, we will firstly show that the set $\mathcal{L}_N(p/N)$ is balanced for every p, N. Let α be an irrational number very close to p/N, then $\mathrm{Fact}_N(\boldsymbol{v}^\omega) = \mathcal{L}_N(p/N) \subseteq \mathcal{L}_N(\alpha) = \mathrm{Fact}_N(\boldsymbol{u})$ for some sturmian word $\boldsymbol{u}$,[2] so the set $\mathrm{Fact}_n(\boldsymbol{u})$ is balanced for every $n \in \mathbb{N}$. But as $\mathrm{Fact}_N(\boldsymbol{v}^\omega) \subseteq \mathrm{Fact}_N(\boldsymbol{u})$, we have $\mathrm{Fact}_n(\boldsymbol{v}^\omega) \subseteq \mathrm{Fact}_n(\boldsymbol{u})$ for every $n \leq N$ hence the set $\mathrm{Fact}_n(\boldsymbol{v}^\omega)$ is balanced.

The word $\boldsymbol{v}^\omega$ is balanced if the set $\mathrm{Fact}_n(\boldsymbol{v}^\omega)$ is balanced for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, take any $k \in \mathbb{N}$ such that $kN \geq n$. Then $\boldsymbol{w} := \boldsymbol{v}^k \in \mathcal{L}_{kN}(p/N) = \mathcal{L}_{kN}(kp/kN)$ and $\boldsymbol{v}^\omega = \boldsymbol{w}^\omega$. Finally, the previous gives that $\mathrm{Fact}_n(\boldsymbol{w}^\omega)$ is a balanced set, proving the claim. □

3.4 Standard word tree

It will be proven that every sturmian morphism is conjugate to some standard morphism. Definition of standard tree and standard morphisms is the topic of this section. The construction of standard word tree is shown in Figure 3.5.

The root node is the pair $(0,1)$. The tree is binary, i.e. every node $(\boldsymbol{u}, \boldsymbol{v})$ has 2 children. The left child node is $(\boldsymbol{u}, \boldsymbol{uv})$ and the right child node is $(\boldsymbol{vu}, \boldsymbol{v})$.

From the construction of standard tree the following properties about matrices of pairs are clear.

Property 3.21. **(1)** *For the head node of the standard tree holds that*

$$\mathbf{M}(0,1) = \mathbf{I}.$$

(2) *For every node $(\boldsymbol{u}, \boldsymbol{v})$ holds that*

$$\mathbf{M}(\boldsymbol{u}, \boldsymbol{uv}) = \mathbf{M}(\boldsymbol{u}, \boldsymbol{v}) \cdot \mathbf{L} \quad \text{and} \quad \mathbf{M}(\boldsymbol{vu}, \boldsymbol{v}) = \mathbf{M}(\boldsymbol{u}, \boldsymbol{v}) \cdot \mathbf{R},$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[2] For each p, N there exists $\varepsilon > 0$ such that all $\alpha \in (p/N - \varepsilon, p/N + \varepsilon)$ satisfy this.

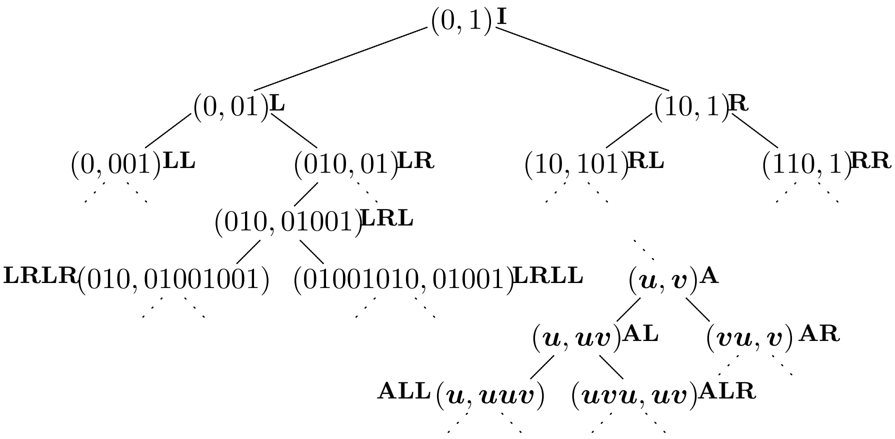

Figure 3.5: Standard word tree

In this section, the results about set $\mathrm{SL}(2,\mathbb{N})$ of all $\mathbb{N}^{2,2}$ matrices with determinant equal to one, which are summarized in Appendix A, are used.

The following theorem allows us to identify the standard tree nodes with the set $\mathrm{SL}(2,\mathbb{N})$.

Theorem 3.22. **(1)** *For each standard pair* $(\boldsymbol{u},\boldsymbol{v})$ *determinant of its matrix is equal to* 1.

(2) *For any non-negative integer* 2×2 *matrix* $\mathbf{A}$ *with determinant equal to one, there exists exactly one standard pair* $(\boldsymbol{u},\boldsymbol{v})$ *such that* $\mathbf{A} = \mathbf{M}(\boldsymbol{u},\boldsymbol{v})$.

Proof. **(1)** Holds that $\det \mathbf{M}(0,1) = 1$, and for every node holds that determinant of its child nodes is the same. Mathematical induction gives us the claim.

(2) According to Lemma A.3, there exists unique factorization of matrix $\mathbf{A}$ by matrices $\mathbf{L}$ and $\mathbf{R}$. Denote n the length of the factorization, for $\mathbf{A} = \mathbf{I}$ put $n = 0$. We prove the claim by mathematical induction on n.

$(n = 0)$ Clearly $\det \mathbf{I} = 1$.

$(n \to n+1)$ Let $\mathbf{A} = \mathbf{BX}$ for $\mathbf{X} \in \{\mathbf{L}, \mathbf{R}\}$. According to induction hypothesis there exists standard pair $(\boldsymbol{u}', \boldsymbol{v}')$ such that $\mathbf{M}(\boldsymbol{u}', \boldsymbol{v}') = \mathbf{B}$. For $\mathbf{X} = \mathbf{L}$ put $\boldsymbol{u} := \boldsymbol{u}'$, $\boldsymbol{v} := \boldsymbol{u}'\boldsymbol{v}'$ and for $\mathbf{X} = \mathbf{R}$ put $\boldsymbol{u} := \boldsymbol{v}'\boldsymbol{u}'$, $\boldsymbol{v} := \boldsymbol{v}'$. Then $(\boldsymbol{u},\boldsymbol{v})$ is a standard pair and $\mathbf{M}(\boldsymbol{u},\boldsymbol{v}) = \mathbf{BX} = \mathbf{A}$.

From the first claim it is clear that finite products of matrices $\mathbf{L}, \mathbf{R}$ cover all matrices of standard pairs, and as the factorization is unique, there exists exactly one standard pair with matrix $\mathbf{A}$. $\square$

According to the Appendix, we can identify standard pairs with $\mathbb{N}^{2,2}$ matrices with determinant equal to 1. Set of all such matrices is a monoid of words on the alphabet $\{\mathbf{L}, \mathbf{R}\}$, which can be in accordance with notation from Chapter 2 denoted by $\{\mathbf{L}, \mathbf{R}\}^*$. The matrix $\mathbf{I}$ represents the empty word in $\{\mathbf{L}, \mathbf{R}\}^*$.

Proposition 3.23. *Let $(\boldsymbol{u}, \boldsymbol{v})$ be a standard pair. Then there exists word $\boldsymbol{s} \in \{0,1\}^*$ such that*

$$\boldsymbol{uv} = \boldsymbol{s}01 \quad \textit{and} \quad \boldsymbol{vu} = \boldsymbol{s}10.$$

Proof. We prove the claim by induction on $|\boldsymbol{uv}|$.

($(0,1)$) For this pair, the claim holds putting $\boldsymbol{s} = \boldsymbol{\varepsilon}$.

($(\boldsymbol{u}, \boldsymbol{v}) \to (\boldsymbol{u}, \boldsymbol{uv})$) Let $\boldsymbol{uv} = \boldsymbol{s}01$ and $\boldsymbol{vu} = \boldsymbol{s}10$. Then $\boldsymbol{u}(\boldsymbol{uv}) = (\boldsymbol{us})01$ and $(\boldsymbol{uv})\boldsymbol{u} = (\boldsymbol{us})10$.

($(\boldsymbol{u}, \boldsymbol{v}) \to (\boldsymbol{vu}, \boldsymbol{v})$) Let $\boldsymbol{uv} = \boldsymbol{s}01$ and $\boldsymbol{vu} = \boldsymbol{s}10$. Then $(\boldsymbol{vu})\boldsymbol{v} = (\boldsymbol{vs})01$ and $\boldsymbol{v}(\boldsymbol{vu}) = (\boldsymbol{vs})10$. $\square$

3.5 Conjugation

Definition 3.24. Let $\boldsymbol{u}, \widetilde{\boldsymbol{u}} \in \mathbb{A}^*$. We say that $\widetilde{\boldsymbol{u}}$ is **right conjugate** to $\boldsymbol{u}$, if there exists $\boldsymbol{s} \in \mathbb{A}^*$ such that $\boldsymbol{us} = \boldsymbol{s}\widetilde{\boldsymbol{u}}$.

Remark. It is clear that the relation of right conjugation is symmetrical on words, so the name "right" might seem to be unnecessary. However, it will be naturally expanded to morphisms, for which the relation is not symmetrical.

Lemma 3.25. *For finite non-empty word $\boldsymbol{u} \in \mathbb{A}^+$ and for $n \in \mathbb{N}$, there exists exactly one $\boldsymbol{s} \in \mathbb{A}^n$ and one $\widetilde{\boldsymbol{u}}$ such that*

$$\boldsymbol{us} = \boldsymbol{s}\widetilde{\boldsymbol{u}}. \tag{3.4}$$

Proof. In the proof, denote $\mathrm{Pref}(\boldsymbol{u})$ the set of all prefixes of the word $\boldsymbol{u}$.

Put $\boldsymbol{s}$ as a prefix of $\boldsymbol{u}^\omega$ such that $|\boldsymbol{s}| = n$. Then $\boldsymbol{s}$ is unique and it is of form $\boldsymbol{u}^k\boldsymbol{t}$ for some $k \in \mathbb{N}$ and $\boldsymbol{t} \in \mathrm{Pref}(\boldsymbol{u})$, where $\mathrm{Pref}(\boldsymbol{u})$ denotes set of all prefixes of $\boldsymbol{u}$. Hence $\boldsymbol{u} = \boldsymbol{tx}$ for some $\boldsymbol{x} \in \mathbb{A}^*$ and $\boldsymbol{us} = \boldsymbol{u}^{k+1}\boldsymbol{t} = \boldsymbol{u}^k\boldsymbol{txt} = \boldsymbol{sxt}$, so we put $\widetilde{\boldsymbol{u}} = \boldsymbol{xt}$.

We show the uniqueness of $(\boldsymbol{s}, \widetilde{\boldsymbol{u}})$. Clearly from (3.4) we have $\widetilde{\boldsymbol{u}}$ given uniquely by $\boldsymbol{u}$ and $\boldsymbol{s}$. Let us suppose that there exists a word $\boldsymbol{s}'$ such that $\boldsymbol{s}' \notin \mathrm{Pref}(\boldsymbol{u}^\omega)$ and take the shortest prefix of $\boldsymbol{s}'$ that is not prefix of $\boldsymbol{u}^\omega$, then it is surely of form $\boldsymbol{u}^\ell \boldsymbol{y} a$ with $\boldsymbol{y} \in \mathrm{Pref}(\boldsymbol{u})$ and $|\boldsymbol{y}| < |\boldsymbol{u}| - 1$. Identity (3.4) gives $\boldsymbol{s}' = \boldsymbol{u}^\ell \boldsymbol{y} a \in \mathrm{Pref}(\boldsymbol{u}^{\ell+1} \boldsymbol{y} a)$, but because $|\boldsymbol{u}| > 0$, holds that $\boldsymbol{u}^\ell \boldsymbol{y} a \in \mathrm{Pref}(\boldsymbol{u}^{\ell+1} \boldsymbol{y}) \subseteq \mathrm{Pref}(\boldsymbol{u}^\omega)$, contradiction. □

Definition 3.26. Let us take $\boldsymbol{u} \in \mathbb{A}^*$ and put $n = 1$ in the previous lemma, we get a word $\widetilde{\boldsymbol{u}}$. Then we write $\widetilde{\boldsymbol{u}} = \mathrm{SHL}(\boldsymbol{u})$.

Property 3.27. *For $\boldsymbol{u} \in \mathbb{A}^+$ holds that $\mathrm{SHL}^{|\boldsymbol{u}|}(\boldsymbol{u}) = \boldsymbol{u}$ and for inverse mapping holds that $\mathrm{SHL}^{-1}(\boldsymbol{u}) = \mathrm{SHL}^{|\boldsymbol{u}|-1}(\boldsymbol{u})$.*

Remark. We call this mapping **Left SHift** and the name comes from computer science.

Proposition 3.28. *Let $p, N \in \mathbb{N}$ be coprime integers such that $p < N$. Then the set $\mathcal{L}_N(p/N)$ is generated by any of its elements and the mapping* SHL*, i.e.*

$$\mathcal{L}_N(p/N) = \left\{\mathrm{SHL}^k(\boldsymbol{u})\right\}_{k \in \mathbb{Z}} \quad \textit{for any } \boldsymbol{u} \in \mathcal{L}_N(p/N).$$

Proof. According to Proposition 3.9 there exists $j \in \widehat{N}$ such that

$$\boldsymbol{u} \sim (T^i_{p/N}(j/N))_{i=0}^{N-1}.$$

Two inclusions will be shown.

($\subseteq$) Let us consider any $\boldsymbol{v} \in \mathcal{L}_N(p/N)$,

$$\boldsymbol{v} \sim (T^i_{p/N}(j'/N))_{i=0}^{N-1} \quad \text{for some } j' \in \widehat{N}.$$

Then clearly

$$\boldsymbol{v} \sim (T^i_{p/N}(j/N))_{i=j'-j}^{j'-j+N-1},$$

which, along with $T^i_{p/N}(j/N) = T^{N+i}_{p/N}(j/N)$, gives $\boldsymbol{v} = \mathrm{SHL}^{j'-j}(\boldsymbol{u})$.

($\supseteq$) Let $k \in \mathbb{Z}$. Then

$$\mathrm{SHL}^k(\boldsymbol{u}) \sim (T^i_{p/N}(j/N))_{i=k}^{k+N-1} = (T^i_{p/N}((j+k)/N))_{i=0}^{N-1},$$

which means that $\mathrm{SHL}^k(\boldsymbol{u}) \in \mathcal{L}_N(p/N)$. □

Definition 3.29. Let φ and ψ be morphisms over $\mathbb{A}$. We say that ψ is **right conjugate** to φ, if exists $\boldsymbol{s} \in \mathbb{A}^*$ such that

$$\varphi(a)\boldsymbol{s} = \boldsymbol{s}\psi(a) \quad \text{for every } a \in \mathbb{A}.$$

Example. Let $\varphi : \begin{matrix} 0 \mapsto 01001 \\ 1 \mapsto 010 \end{matrix}$ and $\psi : \begin{matrix} 0 \mapsto 00101 \\ 1 \mapsto 001 \end{matrix}$. Then putting $\boldsymbol{s} = 01$ proves that ψ is right conjugate to φ.

3.6 Sturmian morphisms

Definition 3.30. Let φ be a morphism over $\{0,1\}$. We say that φ is **sturmian**, if it preserves sturmian words, i.e. if the word $\varphi(\boldsymbol{u})$ is sturmian for any $\boldsymbol{u}$ sturmian.

Property 3.31. *The set of sturmian morphisms is a sub-monoid of all morphisms over* $\{0,1\}$*, i.e.:*

(1) *identity morphism I is sturmian;*

(2) *for any sturmian morphisms φ, ψ, the morphism $\varphi\psi$ is sturmian.*

PROOF. Let $\boldsymbol{u}$ be a sturmian word. Then:

(1) $I(\boldsymbol{u}) = \boldsymbol{u}$ is a sturmian word;

(2) $\psi(\boldsymbol{u})$ is a sturmian word and hence $\varphi(\psi(\boldsymbol{u}))$ is a sturmian word. □

Definition 3.32. Let φ be a morphism over $\mathbb{A}$. We say that φ is **weekly sturmian**, if it preserves one sturmian word, i.e. if exists sturmian word $\boldsymbol{u}$ such that $\varphi(\boldsymbol{u})$ is sturmian.

In the article [2] the following theorem is proven.

Theorem 3.33. *Let φ be a morphism over $\mathbb{A}$. The following claims are equivalent:*

(1) *φ is sturmian;*

(2) *φ is weakly sturmian;*

(3) $\varphi \in \{F, \widetilde{F}\}$

(4) *the word* $\varphi(10010010100101)$ *is balanced.*

The following statement comes from [2], where the proof is not provided.

Lemma 3.34. *Let $\boldsymbol{u} \in \{0,1\}^\infty$ be a word (finite or infinite), which is unbalanced. Then it contains factors of form $0\boldsymbol{t}0$ and $1\boldsymbol{t}1$ such that they are disjoint, i.e. that not any non-empty prefix or suffix of $0\boldsymbol{t}0$ is prefix or suffix of $1\boldsymbol{t}1$.*

PROOF. The word $\boldsymbol{u}$ is unbalanced, therefore it contains factors $\boldsymbol{x}, \boldsymbol{y}$ of the same length such that $\delta(\boldsymbol{x}, \boldsymbol{y}) > 1$. Let us take the shortest $\boldsymbol{x}, \boldsymbol{y}$ with this property, we will prove by contradiction that they have the desired form.

Suppose that $\boldsymbol{x} = \boldsymbol{t}\boldsymbol{x'}$ and $\boldsymbol{y} = \boldsymbol{t}\boldsymbol{y'}$. Hence as $\delta(\boldsymbol{t}, \boldsymbol{t}) = 0$, we have $\delta(\boldsymbol{x'}, \boldsymbol{y'}) = \delta(\boldsymbol{x}, \boldsymbol{y}) > 1$, from whence it follows that $|\boldsymbol{x}| = |\boldsymbol{x'}|$ and $\boldsymbol{t} = \boldsymbol{\varepsilon}$. Similarly we can show the other cases.

Now it is clear that there exist two words $\boldsymbol{t}, \boldsymbol{t'}$ such that $\boldsymbol{x} = a\boldsymbol{t}a$ and $\boldsymbol{y} = b\boldsymbol{t'}b$ as the previous holds for one-letter prefixes and suffixes, particularly. Without loss of generality, suppose $a = 0$ and $b = 1$.

Let us suppose that $\delta(0\boldsymbol{t}0, 1\boldsymbol{t'}1) > 2$. Then $\delta(\boldsymbol{t}0, \boldsymbol{t'}1) > 1$ forming shorter pair, which is in contradiction with minimal length of $\boldsymbol{x}, \boldsymbol{y}$.

So $\delta(0\boldsymbol{t}0, 1\boldsymbol{t'}1) = 2$, hence $\delta(\boldsymbol{t}, \boldsymbol{t}') = 0$ (the case $\delta(\boldsymbol{t}, \boldsymbol{t}') = 4$ is again in contradiction with minimal length of $\boldsymbol{x}, \boldsymbol{y}$).

We finally show that $\boldsymbol{t} = \boldsymbol{t}'$. Let $\boldsymbol{s}$ be the longest common prefix of $\boldsymbol{t}, \boldsymbol{t'}$. In the case that $\boldsymbol{x} = 0\boldsymbol{s}1\boldsymbol{z}0$ and $\boldsymbol{y} = 1\boldsymbol{s}0\boldsymbol{z'}1$ we have $\delta(\boldsymbol{z}0, \boldsymbol{z'}1) = 2$, shorter than $\boldsymbol{x}, \boldsymbol{y}$. In the case that $\boldsymbol{x} = 0\boldsymbol{s}0\boldsymbol{z}0$ and $\boldsymbol{y} = 1\boldsymbol{s}1\boldsymbol{z'}1$ we have $\delta(0\boldsymbol{s}0, 1\boldsymbol{s'}1) = 2$, shorter than $\boldsymbol{x}, \boldsymbol{y}$. That means that $\boldsymbol{s} = \boldsymbol{t} = \boldsymbol{t'}$. □

Definition 3.35. Let φ be a morphism over $\mathbb{A}$. We say that φ is a **standard morphism**, if exists standard pair $(\boldsymbol{u}, \boldsymbol{v})$ such that

$$\varphi : \begin{matrix} 0 \mapsto \boldsymbol{u} \\ 1 \mapsto \boldsymbol{v} \end{matrix} \qquad \text{or} \qquad \varphi : \begin{matrix} 0 \mapsto \boldsymbol{v} \\ 1 \mapsto \boldsymbol{u} \end{matrix}.$$

The following propositions are taken from [3], Propositions 2.3.21–22.

Proposition 3.36. *Let φ be a standard morphism, put $N = |\varphi(01)|$. Then there are exactly $N - 1$ morphisms right conjugate to φ, which means that the longest common prefix of $\varphi(0)^\omega$ and $\varphi(1)^\omega$ comprises $N - 2$ letters.*

Proposition 3.37. *Morphism ψ over $\mathbb{A}$ is sturmian, if and only if, there exists standard morphism φ such that ψ is right conjugate to φ.*

Example. Let us take standard pair $(\boldsymbol{u}, \boldsymbol{v}) = (010, 01001)$, and two standard morphisms

$$\varphi^{[0]} : \begin{matrix} 0 \mapsto \boldsymbol{u} \\ 1 \mapsto \boldsymbol{v} \end{matrix} \quad \text{and} \quad \widetilde{\varphi}^{[0]} : \begin{matrix} 0 \mapsto \boldsymbol{v} \\ 1 \mapsto \boldsymbol{u} \end{matrix}.$$

The process of conjugation is shown in Table 3.2. We take the words $\varphi^{[0]}(01), \widetilde{\varphi}^{[0]}(01)$ as a whole and form a right conjugate words to this concatenated word; there are 8 of them, numbered $0, \ldots, 7$. According to Proposition 3.36, all but last are right conjugate to $\varphi^{[0]}$ and $\widetilde{\varphi}^{[0]}$. As well, we see that $\varphi^{[7]}$ and $\widetilde{\varphi}^{[7]}$ have different matrix than the previous ones.

Lemma 3.38. *For each standard word $\boldsymbol{u}$ holds that $\boldsymbol{u} \in \mathcal{L}_N(p/N)$, where $N = |\boldsymbol{u}|$ and $p = |\boldsymbol{u}|_0$.*

k	$\varphi^{[k]}(0)$	$\varphi^{[k]}(1)$		k	$\widetilde{\varphi}^{[k]}(0)$	$\widetilde{\varphi}^{[k]}(1)$
0	010	01001		0	01001	010
1	100	10010		1	10010	100
2	001	00101		2	00101	001
3	010	01010	conjugate to $\varphi^{[0]}$ and $\widetilde{\varphi}^{[0]}$	3	01010	010
4	100	10100		4	10100	100
5	001	01001		5	01001	001
6	010	10010		6	10010	010
7	101	00100	not conjugate	7	00100	101

Table 3.2: Example of conjugate morphisms to standard morphism.

Proof. We will show that $\boldsymbol{u}^\omega$ is a balanced word, and then, according to Lemma 2.1.15 in [3], it is a 2iet word.

Let φ be a standard morphism such that $\varphi(0) = \boldsymbol{u}$. Let us take two factors $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Fact}_\ell(\boldsymbol{u}^\omega)$ of length $\ell \in \mathbb{N}$. Then surely exists finite $k \in \mathbb{N}$ such that $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Fact}(\boldsymbol{u}^k)$. Let us take a standard word $\boldsymbol{v} = 0^k 1$, and any sturmian word $\boldsymbol{s}$ such that $\boldsymbol{v} \in \mathrm{Fact}(\boldsymbol{s})$. Then $\varphi(\boldsymbol{s})$ is sturmian and $\boldsymbol{u}^k \in \mathrm{Fact}(\varphi(\boldsymbol{s}))$. From this we have that $\boldsymbol{u}^k$ is balanced and $\delta(\boldsymbol{x}, \boldsymbol{y}) \leq 1$.

So $\boldsymbol{u}^\omega$ is a 2iet word, and counting occurrences of 0 and 1 in $\boldsymbol{u}^\omega$ gives

$$\boldsymbol{u}^\omega \sim (T^i_{p/N}(\rho))_{i=0}^{\infty}$$

for some $\rho \in [0, 1)$. □

Proposition 3.39. *Let $(\boldsymbol{u}, \boldsymbol{v})$ be a standard pair, put $N := |\boldsymbol{uv}|$ and $p := |\boldsymbol{uv}|_0$. Then N and p are coprime integers and*

$$\left\{\mathrm{SHL}^k(\boldsymbol{uv})\right\}_{k=0}^{N-1} = \mathcal{L}_N(p/N).$$

Proof. The fact that p and N are coprime follows from $\det \mathbf{M}(\boldsymbol{u}, \boldsymbol{v}) = 1$.

It is easy to see that for a standard pair $(\boldsymbol{u}, \boldsymbol{v})$, the word $\boldsymbol{uv}$ is a standard word. The claim then follows from the previous lemma and from Proposition 3.28. □

Chapter 4

Amicability

Convention. Let $\mathbb{A} = \{a_0, \dots, a_{k-1}\}$ be an alphabet, and let $\mathbb{B} \subsetneq \mathbb{A}$. Let φ be a morphism over $\mathbb{A}$ such that $\varphi(a_0 a_1 \dots a_{k-1}) \in \mathbb{B}^*$. Then we say that φ is a morphism $\mathbb{A} \to \mathbb{B}$.

4.1 Amicable words

Definition 4.1. We define two morphisms $\sigma_{01}, \sigma_{10} : \{0,1,2\}^\infty \to \{0,1\}^\infty$ as follows:

$$\begin{aligned} 0 &\mapsto 0 \\ \sigma_{01} : 1 &\mapsto 1 \\ 2 &\mapsto 01 \end{aligned} \qquad \begin{aligned} 0 &\mapsto 0 \\ \sigma_{10} : 1 &\mapsto 1 \\ 2 &\mapsto 10 \end{aligned} \; .$$

Definition 4.2. Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$. We say that $\boldsymbol{u}$ **is amicable to** $\boldsymbol{v}$ (denoted $\boldsymbol{u} \propto \boldsymbol{v}$), if there exists $\boldsymbol{s} \in \{0,1,2\}^\infty$ such that

$$\boldsymbol{u} = \sigma_{01}(\boldsymbol{s}) \quad \text{and} \quad \boldsymbol{v} = \sigma_{10}(\boldsymbol{s}).$$

The word $\boldsymbol{s}$ is called **ternarization** of $\boldsymbol{u}$ and $\boldsymbol{v}$ and we write $\boldsymbol{s} = \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})$.

Example. Let

$$\begin{aligned} \boldsymbol{u} &= 01\,0\,01\,0\,1\,0\,, \\ \boldsymbol{v} &= 10\,0\,10\,0\,1\,0\,, \\ \boldsymbol{s} &= \;\,2\;\;0\;\;2\;\;0\,1\,0\,. \end{aligned}$$

Then $\boldsymbol{u} \propto \boldsymbol{v}$ and $\mathrm{ter}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{s}$.

Proposition 4.3. *Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^*$ be two words such that $\boldsymbol{u} \propto \boldsymbol{v}$, and let $\boldsymbol{s} \coloneqq \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})$. Then*

$$\begin{aligned} |\boldsymbol{u}|_0 = |\boldsymbol{v}|_0 = |\boldsymbol{s}|_0 + |\boldsymbol{s}|_2\,, \\ |\boldsymbol{u}|_1 = |\boldsymbol{v}|_1 = |\boldsymbol{s}|_1 + |\boldsymbol{s}|_2\,. \end{aligned}$$

Proof. For matrices of morphisms σ_{01}, σ_{10} holds that

$$\mathbf{M}_{\sigma_{01}} = \mathbf{M}_{\sigma_{10}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

which means that

$$\begin{pmatrix} |\boldsymbol{s}|_0 + |\boldsymbol{s}|_2 \\ |\boldsymbol{s}|_1 + |\boldsymbol{s}|_2 \end{pmatrix} = \mathbf{M}\left(\sigma_{01}(\boldsymbol{s})\right) = \mathbf{M}\left(\boldsymbol{u}\right) = \begin{pmatrix} |\boldsymbol{u}|_0 \\ |\boldsymbol{u}|_1 \end{pmatrix},$$

and similarly for σ_{10} and $\boldsymbol{v}$. □

Proposition 4.4. *Let $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^*$ and $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ be words such that $\boldsymbol{x} \propto \boldsymbol{y}$ and $\boldsymbol{u} \propto \boldsymbol{v}$. Then $\boldsymbol{xu} \propto \boldsymbol{yv}$ and*

$$\operatorname{ter}(\boldsymbol{xu}, \boldsymbol{yv}) = \operatorname{ter}(\boldsymbol{x}, \boldsymbol{y}) \operatorname{ter}(\boldsymbol{u}, \boldsymbol{v}).$$

Proof. There exist $\boldsymbol{s} \in \{0,1,2\}^*$ and $\boldsymbol{t} \in \{0,1,2\}^\infty$ such that

$$\begin{aligned} \boldsymbol{x} &= \sigma_{01}(\boldsymbol{s}), & \boldsymbol{y} &= \sigma_{10}(\boldsymbol{s}), \\ \boldsymbol{u} &= \sigma_{01}(\boldsymbol{t}), & \boldsymbol{v} &= \sigma_{10}(\boldsymbol{t}). \end{aligned}$$

Hence $\boldsymbol{xu} = \sigma_{01}(\boldsymbol{st})$ and $\boldsymbol{yv} = \sigma_{10}(\boldsymbol{st})$, which means that $\boldsymbol{xu} \propto \boldsymbol{zw}$ with $\operatorname{ter}(\boldsymbol{xu}, \boldsymbol{yv}) = \boldsymbol{st}$. □

Lemma 4.5. *Put $\boldsymbol{u} = \boldsymbol{v}a\boldsymbol{w}$ and $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{v}}\widetilde{a}\widetilde{\boldsymbol{w}}$ for some words $\boldsymbol{v}, \widetilde{\boldsymbol{v}} \in \{0,1\}^*$ of the same length, for some $\boldsymbol{w}, \widetilde{\boldsymbol{w}} \in \{0,1\}^\infty$ of the same length, and for some letters $a, \widetilde{a} \in \{0,1\}$. Suppose $\boldsymbol{u} \propto \widetilde{\boldsymbol{u}}$. Then*

$$\boldsymbol{v} \propto \widetilde{\boldsymbol{v}} \ \wedge \ a\boldsymbol{w} \propto \widetilde{a}\widetilde{\boldsymbol{w}} \qquad \textit{or} \qquad \boldsymbol{v}a \propto \widetilde{\boldsymbol{v}}\widetilde{a} \ \wedge \ \boldsymbol{w} \propto \widetilde{\boldsymbol{w}}.$$

Proof. If $a = \widetilde{a}$ then $a \propto \widetilde{a}$ and thus $\operatorname{ter}(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) = \operatorname{ter}(\boldsymbol{v}, \widetilde{\boldsymbol{v}}) a \operatorname{ter}(\boldsymbol{w}, \widetilde{\boldsymbol{w}})$.

If $a = 1$ and $\widetilde{a} = 0$, then necessarily $\boldsymbol{v} = \boldsymbol{x}0$ and $\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{x}}1$ for some $\boldsymbol{x}, \widetilde{\boldsymbol{x}}$. And thus $\operatorname{ter}(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) = \operatorname{ter}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) 2 \operatorname{ter}(\boldsymbol{w}, \widetilde{\boldsymbol{w}})$.

If $a = 0$ and $\widetilde{a} = 1$, then $\boldsymbol{w} = 1\boldsymbol{x}$ and $\widetilde{\boldsymbol{w}} = 0\widetilde{\boldsymbol{x}}$, and thus $\operatorname{ter}(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) = \operatorname{ter}(\boldsymbol{v}, \widetilde{\boldsymbol{v}}) 2 \operatorname{ter}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$. □

We would like to reverse the implication in Proposition 4.4. To do so, we need to add one additional condition.

Proposition 4.6. *Let $\boldsymbol{u} = \boldsymbol{vw}$ and $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{v}}\widetilde{\boldsymbol{w}}$ for some words $\boldsymbol{v}, \widetilde{\boldsymbol{v}} \in \{0,1\}^*$ of the same length, and for some $\boldsymbol{w}, \widetilde{\boldsymbol{w}} \in \{0,1\}^\infty$ of the same length. Suppose $\boldsymbol{u} \propto \widetilde{\boldsymbol{u}}$ and $|\boldsymbol{v}|_0 = |\widetilde{\boldsymbol{v}}|_0$. Then*

$$\boldsymbol{v} \propto \widetilde{\boldsymbol{v}} \qquad \textit{and} \qquad \boldsymbol{w} \propto \widetilde{\boldsymbol{w}}.$$

Proof. Put $\boldsymbol{s} \coloneqq \operatorname{ter}(\boldsymbol{u}, \widetilde{\boldsymbol{u}})$. Take $\boldsymbol{x}$ the longest prefix of $\boldsymbol{s}$ such that $\sigma_{01}(\boldsymbol{x})$ is a prefix of $\boldsymbol{v}$. Then either $\boldsymbol{v} = \sigma_{01}(\boldsymbol{x})$ or $\boldsymbol{v} = \sigma_{01}(\boldsymbol{x})a$ for some letter $a \in \{0,1\}$.

If the first holds, then as well $\widetilde{\boldsymbol{v}} = \sigma_{10}(\boldsymbol{x})$ and $\boldsymbol{v} \propto \widetilde{\boldsymbol{v}}$.

If the second holds, then $\widetilde{\boldsymbol{v}} = \sigma_{10}(\boldsymbol{x})\widetilde{a}$ and $a \neq \widetilde{a}$ (because if $a = \widetilde{a}$, then $\sigma_{01}(\boldsymbol{x}a) = \boldsymbol{v}$ and $\boldsymbol{x}a$ is a prefix of $\boldsymbol{s}$ longer than $\boldsymbol{x}$). But $|\sigma_{01}(\boldsymbol{x})|_0 = |\sigma_{10}(\boldsymbol{x})|_0$ and $a \neq \widetilde{a}$, thus $|\sigma_{01}(\boldsymbol{x})a|_0 \neq |\sigma_{10}(\boldsymbol{x})\widetilde{a}|_0$, which is in contradiction with $\boldsymbol{v} \propto \widetilde{\boldsymbol{v}}$.

Now as $\boldsymbol{x}$ is a prefix of $\boldsymbol{s}$, there exists $\boldsymbol{y} \in \{0,1\}^\infty$ such that $\boldsymbol{s} = \boldsymbol{x}\boldsymbol{y}$, hence $\boldsymbol{w} = \sigma_{01}(\boldsymbol{y})$ and $\widetilde{\boldsymbol{w}} = \sigma_{10}(\boldsymbol{y})$, giving $\boldsymbol{w} \propto \widetilde{\boldsymbol{w}}$. □

Proposition 4.7. *Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ be two words. Then*

$$\boldsymbol{u} \propto \boldsymbol{v} \iff E(\boldsymbol{v}) \propto E(\boldsymbol{u}).$$

Proof. For purposes of this proof, we expand the morphism $E : \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix}$ over $\{0,1\}$ to morphism D over $\{0,1,2\}$ given as $D : \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0. \\ 2 \mapsto 2 \end{matrix}$

(⇒) Holds that $E\sigma_{01}D = \sigma_{10}$ and $E\sigma_{10}D = \sigma_{01}$. Put $\boldsymbol{s} = \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})$. Then $\sigma_{01}(D(\boldsymbol{s})) = E^2\sigma_{01}D(\boldsymbol{s}) = E\sigma_{10}(\boldsymbol{s}) = E(\boldsymbol{v})$ and similarly $\sigma_{10}(D(\boldsymbol{s})) = E(\boldsymbol{u})$, which means that $D(\boldsymbol{s}) = \mathrm{ter}(E(\boldsymbol{v}), E(\boldsymbol{u}))$ and $E(\boldsymbol{v}) \propto E(\boldsymbol{u})$.

(⇐) Applying the first implication on the right side we get $\boldsymbol{u} = E^2(\boldsymbol{u}) \propto E^2(\boldsymbol{v}) = \boldsymbol{v}$. □

Proposition 4.8. *Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ be two words. Then*

$$\boldsymbol{u} \propto \boldsymbol{v} \iff \boldsymbol{v}^{\mathsf{R}} \propto \boldsymbol{u}^{\mathsf{R}}.$$

Proof. Let $\boldsymbol{x} \in \{0,1,2\}^\infty$. Then clearly $\sigma_{01}(\boldsymbol{x})^{\mathsf{R}} = \sigma_{10}\left(\boldsymbol{x}^{\mathsf{R}}\right)$ and vice versa.

Now let $\boldsymbol{s} := \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})$. Then $\boldsymbol{v}^{\mathsf{R}} = \sigma_{10}(\boldsymbol{s})^{\mathsf{R}} = \sigma_{01}\left(\boldsymbol{s}^{\mathsf{R}}\right)$ and $\boldsymbol{u}^{\mathsf{R}} = \sigma_{01}(\boldsymbol{s})^{\mathsf{R}} = \sigma_{10}\left(\boldsymbol{s}^{\mathsf{R}}\right)$, which gives $\boldsymbol{v}^{\mathsf{R}} \propto \boldsymbol{u}^{\mathsf{R}}$ and $\mathrm{ter}(\boldsymbol{v}^{\mathsf{R}}, \boldsymbol{u}^{\mathsf{R}}) = \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})^{\mathsf{R}}$. □

Proposition 4.9. *Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ be two words such that $\boldsymbol{u} \propto \boldsymbol{v}$. Then $\boldsymbol{u} \preceq \boldsymbol{v}$.*

Proof. Put $\boldsymbol{s} := \mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})$. If $|\boldsymbol{s}|_2 = 0$ then $\boldsymbol{u} = \boldsymbol{v}$.

If $|\boldsymbol{s}|_2 \geq 1$, then we factorize the word $\boldsymbol{s}$ by the first 2, so we have $\boldsymbol{s} = \boldsymbol{w}2\boldsymbol{z}$, where $|\boldsymbol{w}|_2 = 0$. Thus $\boldsymbol{u} = \boldsymbol{w}01\sigma_{01}(\boldsymbol{z})$ and $\boldsymbol{w}10\sigma_{10}(\boldsymbol{z}) = \boldsymbol{v}$, giving the claim. □

Corollary 4.10. *Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ be two words of the same length. Then*

$$\boldsymbol{u} \succ \boldsymbol{v} \implies \boldsymbol{u} \not\propto \boldsymbol{v}.$$

Definition 4.11. Let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^*$ be finite words such that $\boldsymbol{u} \propto \boldsymbol{v}$. Put

$$b := |\mathrm{ter}(\boldsymbol{u}, \boldsymbol{v})|_2 .$$

Then we say that $\boldsymbol{u}$ is b**-amicable** to $\boldsymbol{v}$.

Proposition 4.12. *For any standard pair $(\boldsymbol{u}, \boldsymbol{v})$, the word $\boldsymbol{uv}$ is 1-amicable to the word $\boldsymbol{vu}$.*

Proof. We know that there exists $\boldsymbol{s} \in \{0,1\}^*$ such that $\boldsymbol{uv} = \boldsymbol{s}01$ and $\boldsymbol{vu} = \boldsymbol{s}10$. This gives the claim and we have $\mathrm{ter}(\boldsymbol{uv}, \boldsymbol{vu}) = \boldsymbol{s}2$. □

4.2 Amicable morphisms

We want to naturally expand the relation amicability from words to morphisms. The proposition will give us reason why is the definition is natural.

Definition 4.13. Let φ and ψ be morphisms over $\{0,1\}$. We say that φ **is amicable to** ψ (denoted $\varphi \propto \psi$) if:

(1) $\varphi(0) \propto \psi(0)$;

(2) $\varphi(1) \propto \psi(1)$;

(3) $\varphi(01) \propto \psi(10)$.

We define $\mathrm{ter}(\varphi, \psi)$ as morphism $\{0,1,2\}^\infty \to \{0,1,2\}^\infty$ given by

$$\mathrm{ter}(\varphi,\psi) : \begin{aligned} 0 &\mapsto \mathrm{ter}(\varphi(0), \psi(0)) \\ 1 &\mapsto \mathrm{ter}(\varphi(1), \psi(1)) \\ 2 &\mapsto \mathrm{ter}(\varphi(01), \psi(10)) \end{aligned}$$

and we call it **ternarization** of morphisms φ, ψ.

Proposition 4.14. *Let φ and ψ be morphisms over $\{0,1\}$ such that $\varphi \propto \psi$ and let $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty$ such that $\boldsymbol{u} \propto \boldsymbol{v}$. Then*

$$\varphi(\boldsymbol{u}) \propto \psi(\boldsymbol{v}).$$

Proof. As (ter) and $(\cdot)$ commute on words we can show just for $(\boldsymbol{u}, \boldsymbol{v}) \in \{(0,0), (1,1), (01,10)\}$ and then the claim holds from the fact, that every pair of amicable words is generated by 3 named pairs. But the claim for 3 mentioned pairs is just the definition of amicability of morphisms. □

Lemma 4.15. *Let φ, ψ be morphisms over $\{0,1\}$. Then*

$$\varphi \propto \psi \qquad \textit{if and only if} \qquad (\forall \boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^\infty)(\boldsymbol{u} \propto \boldsymbol{v} \implies \varphi(\boldsymbol{u}) \propto \psi(\boldsymbol{v})).$$

PROOF. (⇒) This implication is the claim of previous proposition.

(⇐) Taking $(\boldsymbol{u}, \boldsymbol{v}) \in \{(0,0),(1,1),(01,10)\}$ leads to definition of amicability of morphisms.

□

Proposition 4.16. *Let φ, ψ, μ, ν be morphisms over $\{0,1\}$ such that $\varphi \propto \psi$ and $\mu \propto \nu$. Then $\mu\varphi \propto \nu\psi$.*

PROOF. Take any $\boldsymbol{u} \propto \boldsymbol{v}$. Then $\varphi(\boldsymbol{u}) \propto \psi(\boldsymbol{v})$ and $\mu(\varphi(\boldsymbol{u})) \propto \nu(\psi(\boldsymbol{v}))$. And as we took any amicable pair $\boldsymbol{u}, \boldsymbol{v}$, we get $\mu\varphi \propto \nu\psi$. □

Proposition 4.17. *Let φ, ψ be morphisms over $\{0,1\}$. Then*

$$\varphi \propto \psi \qquad \textit{if and only if} \qquad E\psi E \propto E\varphi E.$$

PROOF. (⇒) From amicability $\varphi \propto \psi$ we have

$$\begin{aligned} E\psi E(0) = E\psi(1) \propto E\varphi(1) = E\varphi E(0), \\ E\psi E(1) = E\psi(0) \propto E\varphi(0) = E\varphi E(1), \\ E\psi E(01) = E\psi(10) \propto E\varphi(01) = E\varphi E(10). \end{aligned}$$

(⇐) If we apply the first implication on the right side, we get the claim. □

Definition 4.18. Let φ, ψ be morphisms over $\{0,1\}$ such that $\varphi \propto \psi$. Put

$$b := \big|\mathrm{ter}\big(\varphi(01), \psi(10)\big)\big|_2 .$$

Then we say that φ is **b-amicable** to ψ.

Chapter 5

Amicable Pairs of Sturmian Morphisms

Lemma 5.1. *Let $p, N \in \mathbb{N}$ be coprime numbers such that $0 < p < N$, put $m := \min\{p, N-p\}$ and let $b \in \mathbb{N}$. Then*

$$\textit{number of } b\textit{-amicable pairs in } \mathcal{L}_N(p/N) = \begin{cases} N-b & \textit{if} \quad 0 \le b \le m \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. We will shorten $T := T_{p/N}$.

Let us suppose that $p < N/2$, thus $m = p$. For $p > N/2$ see comment in the end of the proof.

According to Proposition 3.9 for the set $\mathcal{L}_N(p/N)$ holds that

$$\mathcal{L}_N(p/N) = \{\boldsymbol{w}^{(j)}\}_{j=0}^{N-1},$$

where

$$\boldsymbol{w}^{(j)} \sim (T^k(j/N))_{k=0}^{N-1}.$$

Remember that for the letters of word $\boldsymbol{w}^{(j)}$ we have

$$w_k^{(j)} = \begin{cases} 0 & \text{if} \quad T^k(\rho_j) < p/N \\ 1 & \text{if} \quad T^k(\rho_j) \ge p/N \end{cases}.$$

Consider $\boldsymbol{w}^{(i)}$, $\boldsymbol{w}^{(j)}$ for any $i, j \in \widehat{N}$ and find, for which i, j the words are b-amicable. The following cases are discussed.

$(j < i)$ We know that $\boldsymbol{w}^{(i)} \succ \boldsymbol{w}^{(j)}$ and according to Corollary 4.10 holds that $\boldsymbol{w}^{(i)} \not\propto \boldsymbol{w}^{(j)}$.

$(j = i)$ Every word is 0-amicable to itself.

$(j = i+1)$ We show that $\boldsymbol{w}^{(i)}$ is 1-amicable to $\boldsymbol{w}^{(j)}$.

For $\boldsymbol{w}^{(i)}$ and $\boldsymbol{w}^{(i+1)}$, there exactly once appears situation shown in Figure 5.1 in the first row—exists one $k \in \widehat{N}$ such that $w_k^{(i)} = 0$ and $w_k^{(i+1)} = 1$.

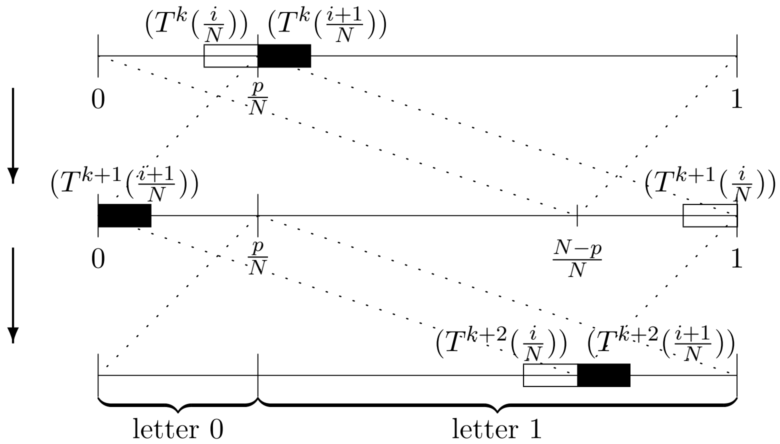

Figure 5.1: To the proof of Lemma 5.1 for $j = i + 1$.

We know that $k \le N-1$ and let us show that $k \le N-2$. Suppose $k = N-1$, then $0 = w^{(i)}_{N-1}$ codes $T^{N-1}(i/N)$ and we have $T\big(T^{N-1}(i/N)\big) = (N-1)/N$ (see Figure 5.1). But $T\big(T^{N-1}(i/N)\big) = T^N(i/N) = i/N$. Hence $i = N-1$ and as $j > i$, holds that $j \nleq N-1$, contradiction.

Now we have $w^{(i)}_{k+1} = 1$ and $w^{(i+1)}_{k+1} = 0$. As the situation in Figure 5.1 appears exactly once for words $\boldsymbol{w}^{(i)}$ and $\boldsymbol{w}^{(i+1)}$, these words can be written as $\boldsymbol{w}^{(i)} = \boldsymbol{x}01\boldsymbol{y}$ and $\boldsymbol{w}^{(j)} = \boldsymbol{w}^{(i+1)} = \boldsymbol{x}10\boldsymbol{y}$, so they are 1-amicable.

$\big(j \in \{i+2, \dots, i+p\}\big)$ Let us find the set I_0 of indexes k such that the words $\boldsymbol{w}^{(i)}, \boldsymbol{w}^{(j)}$ satisfy

$$w^{(i)}_k = 0 \quad \text{and} \quad w^{(j)}_k = 1. \tag{5.1}$$

These k satisfy $T^k(i/N) \in \big[0, \frac{p}{N}\big)$ and $T^k(j/N) \in \big[\frac{p}{N}, 1\big)$. But (see Figure 5.2) such points satisfy $T^k(j/N) = T^k(i/N) + (j-i)/N$. Then (5.1) holds when $T^k(i/N) \in \Big[\frac{p-(j-i)}{N}, \frac{p}{N}\Big)$. This case occurs exactly for $j-i$ different indexes k, hence $\# I_0 = j - i =: b$.

Let us take any $k \in I_0$. We have shown that $w^{(i)}_k = 0$ and $w^{(j)}_k = 1$. Now we need to show that $k \ne N-1$ and $w^{(i)}_{k+1} = 1$ and $w^{(j)}_{k+1} = 0$.

We will show that $k \le N-2$. Suppose $k = N-1$. It holds that $T\big(T^{N-1}(i/N)\big) = T^N(i/N) = i/N \in T^{-1}\big(\big[0, \frac{p}{N}\big)\big) = \Big[\frac{N-p}{N}, 1\Big)$. Hence $i \ge N-p$ and as $j \ge i+p$, holds that $j \nleq N-1$, contradiction.

As $k \le N-2$, we can compute values $w^{(i)}_{k+1}$ and $w^{(j)}_{k+1}$. The situation is shown in Figure 5.2 in the second row, surely $T^{k+1}(i/N) \in \Big[\frac{N-p}{N}, 1\Big)$ hence $w^{(i)}_{k+1} = 1$ and $T^{k+1}(j/N) \in \big[0, \frac{p}{N}\big)$ hence $w^{(j)}_{k+1} = 0$.

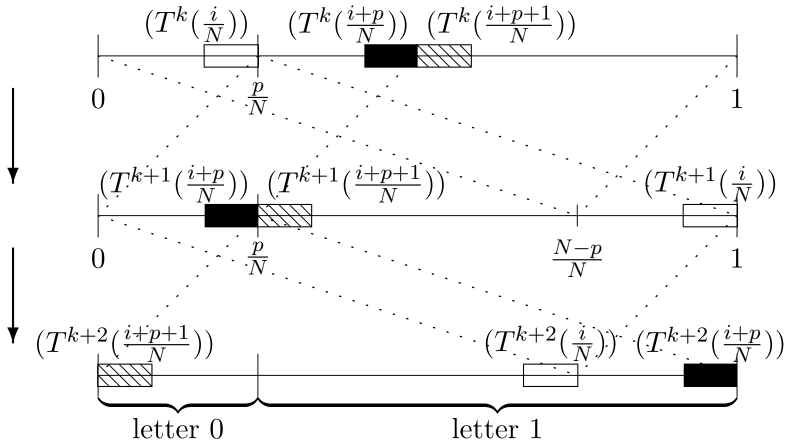

Figure 5.2: To the proof of Lemma 5.1 for $j = i + p$ and $j = i + p + 1$.

So far, we found b occurrences of factors 01 and 10 in $\boldsymbol{w}^{(i)}$ and $\boldsymbol{w}^{(j)}$ on the same positions. We need to explain why the words are equal on all other positions. From $\left|\boldsymbol{w}^{(i)}\right|_0 = \left|\boldsymbol{w}^{(j)}\right|_0$ we know that $\# I_0 = \# I_1$ where $I_1 := \left\{k \in \widehat{N} \middle| w_k^{(i)} = 1 \text{ and } w_k^{(j)} = 0\right\}$. But we know that for every $k \in I_0$, $k+1 \in I_1$. From whence it follows that $I_1 = \left\{k \in \widehat{N} \middle| k-1 \in I_0\right\}$.

Summarized, we shown that words $\boldsymbol{w}^{(i)}$ and $\boldsymbol{w}^{(j)}$ differ only in blocks 01 and 10 and number of these blocks is $b = j-i$, which means that $\boldsymbol{w}^{(i)}$ is b-amicable to $\boldsymbol{w}^{(j)}$. Finally, number of pairs of indexes $(i, j) \in \widehat{N} \times \widehat{N}$ such that $j - i = b$ is exactly $N - b$.

($j \in \{i+p+1, \dots\}$) See again the Figure 5.2, where the situation is shown for $j = i+p+1$. There exists k such that $T^k(i/N) = (p-1)/N$ and $T^k((i+p+1)/N) = 2p/N$ (first row in the figure).

If $k = N-1$, then $\boldsymbol{w}^{(i)}$ ends by 0 and $\boldsymbol{w}^{(j)}$ ends by 1, hence $\boldsymbol{w}^{(i)} \not\propto \boldsymbol{w}^{(j)}$.

If $k \leq N-2$, then $w_k^{(i)} w_{k+1}^{(i)} = 01$ and $w_k^{(j)} w_{k+1}^{(j)} = 11$, hence as well $\boldsymbol{w}^{(i)} \not\propto \boldsymbol{w}^{(j)}$.

Let now $p > N/2$. Propositions 4.7 and 4.8 give $\boldsymbol{u} \propto \boldsymbol{v} \Leftrightarrow E(\boldsymbol{u})^{\mathsf{R}} \propto E(\boldsymbol{v})^{\mathsf{R}}$. Moreover, $\boldsymbol{u} \in \mathcal{L}_N(p/N) \Leftrightarrow E(\boldsymbol{u})^{\mathsf{R}} \in \mathcal{L}_N(1-p/N)$. From this is clear that numbers of b-amicable pairs in $\mathcal{L}_N(p/N)$ and $\mathcal{L}_N(1-p/N)$ are equal. $\square$

Example. Let us explore the previous claims for $p = 3$ and $N = 8$. The set of words $\mathcal{L}_8(3/8)$ is shown in Table 5.1. The occurrences of 01, 10 in consequent words are emphasized by the double line, and it is clearly seen that the consequent words are 1-amicable.

i	$\boldsymbol{w}^{(i)}$	i	$\boldsymbol{w}'^{(i)}$
	01234567		01234567
0	01011011	0	00100101
1	01101011	1	00101001
2	01101101	2	01001001
3	10101101	3	01001010
4	10110101	4	01010010
5	10110110	5	10010010
6	11010110	6	10010100
7	11011010	7	10100100

Table 5.1: The sets of words $\boldsymbol{w}^{(i)} \in \mathcal{L}_8(3/8)$ and $\boldsymbol{w}'^{(i)} \in \mathcal{L}_8(5/8)$

As well, we see that for instance $\boldsymbol{w}^{(1)}$ is 3-amicable to $\boldsymbol{w}^{(4)}$, while 01–10 pairs are on positions 56, 01 and 34.

More, the correspondence between $\mathcal{L}_N(p/N)$ and $\mathcal{L}_N(1-p/N)$ is shown, holds that $E(\boldsymbol{w}^{(i)})^{\mathsf{R}} = \boldsymbol{w}'^{(i)}$.

Theorem 5.2. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & p_1 \\ q_0 & q_1 \end{smallmatrix}\right)$ *be a* $\mathbb{N}^{2,2}$ *matrix such that* $\det \mathbf{A} = p_0q_1 - q_0p_1 = \pm 1$. *Put* $p := p_0 + p_1$, $N := p_0 + p_1 + q_0 + q_1$ *and* $m := \min\{p, N-p\}$. *Let* $b \in \mathbb{N}$. *Then number of pairs of* b*-amicable sturmian morphisms with matrix* $\mathbf{A}$ *is*

$$\textit{for } \det \mathbf{A} = +1 \qquad \begin{cases} N - b & \textit{if} \quad 1 \le b \le m \\ 0 & \textit{otherwise} \end{cases}$$

$$\textit{and for } \det \mathbf{A} = -1 \qquad \begin{cases} N - b - 2 & \textit{if} \quad 0 \le b \le m-1 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. For every $k \in \widehat{N}$ put

$$\boldsymbol{z}^{[k]} := \mathrm{SHL}^k(\boldsymbol{uv}),$$
$$\widetilde{\boldsymbol{z}}^{[k]} := \mathrm{SHL}^k(\boldsymbol{vu}),$$

let $\boldsymbol{u}^{[k]}, \boldsymbol{v}^{[k]}$ be words such that $\left|\boldsymbol{u}^{[k]}\right| = |\boldsymbol{u}|$, $\left|\boldsymbol{v}^{[k]}\right| = |\boldsymbol{v}|$ and

$$\boldsymbol{u}^{[k]}\boldsymbol{v}^{[k]} = \boldsymbol{z}^{[k]},$$

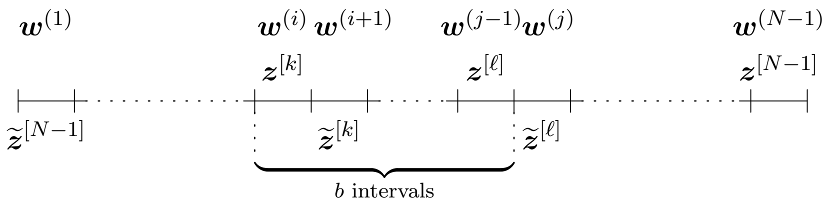

Figure 5.3: To the proof of Theorem 5.2 for $\det \mathbf{A} = +1$.

and define morphisms

$$\varphi^{[k]} : \begin{array}{l} 0 \mapsto \boldsymbol{u}^{[k]} \\ 1 \mapsto \boldsymbol{v}^{[k]} \end{array}$$

$$\widetilde{\varphi}^{[k]} : \begin{array}{l} 0 \mapsto \boldsymbol{v}^{[k]} \\ 1 \mapsto \boldsymbol{u}^{[k]} \end{array}.$$

From Propositions 3.36 and 3.37 we know that for $k \in \{0, \dots, N-2\}$, morphism $\varphi^{[k]}$ is sturmian, but morphism $\varphi^{[N-1]}$ is not.

For $k \in \widehat{N-1}$ holds as well

$$\begin{array}{lll} \boldsymbol{z}^{[k]} = \boldsymbol{u}^{[k]}\boldsymbol{v}^{[k]}, & \varphi^{[k]}(01) = \boldsymbol{z}^{[k]}, & \widetilde{\varphi}^{[k]}(01) = \widetilde{\boldsymbol{z}}^{[k]}, \\ \widetilde{\boldsymbol{z}}^{[k]} = \boldsymbol{v}^{[k]}\boldsymbol{u}^{[k]}, & \varphi^{[k]}(10) = \widetilde{\boldsymbol{z}}^{[k]}, & \widetilde{\varphi}^{[k]}(10) = \boldsymbol{z}^{[k]}. \end{array} \tag{5.2}$$

According to Proposition 3.39 holds

$$\left\{\boldsymbol{z}^{[k]}\right\}_{k=0}^{N-1} = \mathcal{L}_N(p/N) = \left\{\boldsymbol{w}^{(i)}\right\}_{i=0}^{N-1},$$

hence there exists one-to-one mapping $\boldsymbol{z}^{[k]} \leftrightarrow \boldsymbol{w}^{(i)}$ (we use notation $\boldsymbol{w}^{(i)}$ from the proof of Lemma 5.1). But as well, $\widetilde{\boldsymbol{z}}^{[\ell]} = \boldsymbol{z}^{[k]}$ where $\ell \equiv k + |\boldsymbol{u}|$ (mod N), thus there exist one-to-one mappings

$$\widetilde{\boldsymbol{z}}^{[\ell]} \longleftrightarrow \boldsymbol{z}^{[k]} \longleftrightarrow \boldsymbol{w}^{(i)} \longleftrightarrow \widetilde{\boldsymbol{z}}^{[\ell]}.$$

From Proposition 3.23 we know that there exists $\boldsymbol{s}$ such that $\boldsymbol{z}^{[0]} = \boldsymbol{s}01$ and $\widetilde{\boldsymbol{z}}^{[0]} = \boldsymbol{s}10$, hence $\boldsymbol{z}^{[0]} \propto \widetilde{\boldsymbol{z}}^{[0]}$. For any $k \in \widehat{N-1}$ there exist $\boldsymbol{s}', \boldsymbol{s}''$ such that $\boldsymbol{z}^{[k]} = \boldsymbol{s}'01\boldsymbol{s}''$ and $\widetilde{\boldsymbol{z}}k = \boldsymbol{s}'10\boldsymbol{s}''$, hence $\boldsymbol{z}^{[k]}$ is 1-amicable to $\widetilde{\boldsymbol{z}}^{[k]}$. But for $k = N-1$ holds $\boldsymbol{z}^{[N-1]} = 1\boldsymbol{s}0$ and $\widetilde{\boldsymbol{z}}^{[N-1]} = 0\boldsymbol{s}1$, hence $\boldsymbol{z}^{[N-1]} \not\propto \widetilde{\boldsymbol{z}}^{[N-1]}$.

Now, as the only word in $\mathcal{L}_N(p/N)$ which does not have a 1-amicable counterpart is $\boldsymbol{w}^{(N-1)}$, we have $\boldsymbol{z}^{[N-1]} = \boldsymbol{w}^{(N-1)}$. Moreover, as $\boldsymbol{z}^{[k]}$ is 1-amicable to $\widetilde{\boldsymbol{z}}^{[k]}$ for $k \in \widehat{N-1}$, we have that if $\boldsymbol{z}^{[k]} = \boldsymbol{w}^{(i)}$ then $\widetilde{\boldsymbol{z}}^{[k]} = \boldsymbol{w}^{(i+1)}$.

Let us do the proof separately for each determinant.

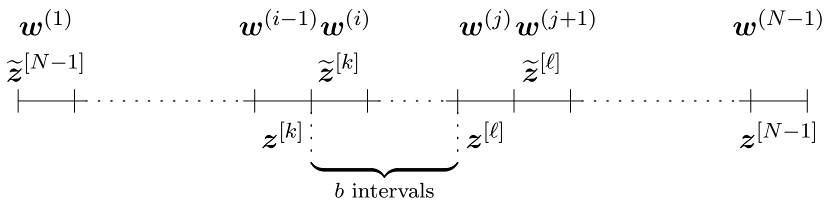

Figure 5.4: To the proof of Theorem 5.2 for $\det \mathbf{A} = -1$.

($\det \mathbf{A} = +1$) Let $i, j \in \widehat{N}$ and $b \in \{1, \dots, m\}$ be numbers such that $j - i = b$, and let $k, \ell \in \widehat{N-1}$ be indexes such that

$$\begin{aligned} \boldsymbol{w}^{(i)} &= \boldsymbol{z}^{[k]} = \varphi^{[k]}(01) = \varphi^{[k]}(0)\varphi^{[k]}(1) \\ \text{and} \quad \boldsymbol{w}^{(j)} &= \widetilde{\boldsymbol{z}}^{[\ell]} = \varphi^{[\ell]}(10) \end{aligned}$$

(see Figure 5.3). Then $\varphi^{[k]}(01)$ is b-amicable to $\widetilde{\boldsymbol{z}}^{[\ell]} = \varphi^{[\ell]}(10)$. Moreover,

$$\boldsymbol{w}^{(j-1)} = \boldsymbol{z}^{[\ell]} = \varphi^{[\ell]}(01) = \varphi^{[\ell]}(0)\varphi^{[\ell]}(1).$$

and $\boldsymbol{w}^{(i)}$ is $(b-1)$-amicable to $\boldsymbol{w}^{(j-1)}$, As well, from the conjugation of $\varphi^{[k]}$ and $\varphi^{[\ell]}$ we know that $\left|\varphi^{[k]}(0)\right|_0 = \left|\varphi^{[\ell]}(0)\right|_0$. Proposition 4.6 then gives $\varphi^{[k]}(0) \propto \varphi^{[\ell]}(0)$ and $\varphi^{[k]}(1) \propto \varphi^{[\ell]}(1)$.

From the proof it is clear that there are not any other amicable pairs. Because when $\varphi^{[k]} \propto \varphi^{[\ell]}$, holds that $\boldsymbol{z}^{[k]} \propto \widetilde{\boldsymbol{z}}^{[\ell]}$ and the amicability is covered by Lemma 5.1.

The number of b-amicable morphisms follows from the lemma. Just for $b = 0$ there are no 0-amicable morphisms, because if $\varphi^{[k]}(01)$ is 0-amicable to $\varphi^{[\ell]}(10)$, then $\varphi^{[k]}(01) \not\propto \varphi^{[\ell]}(01)$.

($\det \mathbf{A} = -1$) Let $i, j \in \widehat{N}$ and $b \in \{0, \dots, m-1\}$ be numbers such that $j - i = b$, and let $k, \ell \in \widehat{N-1}$ be indexes such that

$$\begin{aligned} \boldsymbol{w}^{(i)} &= \widetilde{\boldsymbol{z}}^{[k]} = \widetilde{\varphi}^{[k]}(01), \\ \text{and} \quad \boldsymbol{w}^{(j)} &= \boldsymbol{z}^{[\ell]} = \widetilde{\varphi}^{[\ell]}(10) \end{aligned}$$

(see Figure 5.4). Then $\widetilde{\varphi}^{[k]}(01)$ is b-amicable to $\widetilde{\varphi}^{[\ell]}(10)$. Moreover,

$$\boldsymbol{w}^{(j+1)} = \widetilde{\boldsymbol{z}}^{[\ell]} = \widetilde{\varphi}^{[\ell]}(01),$$

and $\boldsymbol{w}^{(i)}$ is $(b+1)$-amicable to $\boldsymbol{w}^{(j+1)}$, which, along with $\left|\widetilde{\varphi}^{[k]}(0)\right|_0 = \left|\widetilde{\varphi}^{[\ell]}(0)\right|_0$, gives $\widetilde{\varphi}^{[k]}(0) \propto \widetilde{\varphi}^{[\ell]}(0)$ and $\widetilde{\varphi}^{[k]}(1) \propto \widetilde{\varphi}^{[\ell]}(1)$.

From the last it is clear, why $b = m$ disallows $\widetilde{\varphi}^{[k]} \propto \widetilde{\varphi}^{[\ell]}$. As well, we can see that indexes i and j must satisfy $i \geq 2$ and $j \leq N - 2$. These conditions along with the condition $j - i = b$ gives number of pairs of indexes equal to $N - b - 2$. $\square$

Theorem 5.3. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & p_1 \\ q_0 & q_1 \end{smallmatrix}\right)$ *be* $\mathbb{N}^{2,2}$ *matrix such that* $\det \mathbf{A} = \pm 1$*. Put* $p := p_0 + p_1$*,* $N := p_0 + p_1 + q_0 + q_1$ *and* $m := \min\{p, N - p\}$*. Then the number of pairs of amicable sturmian morphisms with matrix* $\mathbf{A}$ *is equal to*

$$m(N-1) + \frac{m}{2}\big(\det \mathbf{A} - m\big).$$

PROOF. Holds that

$$\sum_{b=1}^{m} (N - b) = m(N-1) + \frac{m}{2}\big(1 - m\big)$$

and

$$\sum_{b=0}^{m-1} (N - b - 2) = m(N-1) + \frac{m}{2}\big(-1 - m\big). \qquad \square$$

Conjecture 5.4. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & p_1 \\ q_0 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2,2}$ *be matrix such that* $\det \mathbf{A} = +1$*. Put* $p := p_0 + p_1$*,* $q := q_0 + q_1$*,* $N := p + q$*,* $m := \min\{p, q\}$*. Let* $b \in \{1, \ldots, m\}$ *and let* φ *and* ψ *be sturmian morphisms such that* φ *is* b*-amicable to* ψ *and their matrix is* $\mathbf{A}$*.*

Then for the matrix of morphism $\eta := \operatorname{ter}(\varphi, \psi)$ *holds*

$$\mathbf{M}_\eta = \begin{pmatrix} p_0 - b_0 & p_1 - b_1 & p - b \\ q_0 - b_0 & q_1 - b_1 & q - b \\ b_0 & b_1 & b \end{pmatrix},$$

where

$$b_0 = \lfloor \beta \rfloor \quad \textit{or} \quad b_0 = \lceil \beta \rceil, \qquad \textit{and} \qquad b_1 = (b-1) - b_0$$

for

$$\beta = \frac{p_0 + q_0}{N}\big(b - 1\big).$$

IDEA OF THE CONJECTURE. From the proof of previous theorems we know that $b_0 + b_1 = |\eta(0)|_2 + |\eta(1)|_2 = b - 1$. The conjecture is based on the assumption that the ratio $|\eta(0)|_2 : |\eta(1)|_2$ is similar to ratio $|\eta(0)| : |\eta(1)|$.

The number β is such number that the ratios satisfy

$$\beta : (p_0 + q_0) = (b - 1 - \beta) : (p_1 + q_1) = (b - 1) : N.$$

Then, b_0 and b_0' are the closest integers to β. The other elements of the 3×3 matrix are given uniquely by matrix $\mathbf{A}$ and numbers b, b_0.

By computer numeration, it was found that the conjecture is in accordance with the results in [4], where particular matrices were proven to be or not to be matrices of ternarizations.

Chapter 6

Summary

6.1 Results

We have summarized definitions and properties of finite and infinite words and equivalent definitions of sturmian words, as well as properties of morphisms that preserve the set of sturmian words.

Our most important results are in Section 5, where the main task is solved in two theorems, Theorem 5.2 and Theorem 5.3. We found numbers of pairs of amicable sturmian morphisms for every matrix.

6.2 Further work

using numerical experiments, we have stated a conjecture, proof of which is in the scope of the further work on the topic. Generally, for every matrix $\mathbf{A}$ with $\det \mathbf{A} = \pm 1$, we would like to find matrices of ternarizations, and for each matrix, find number of ternarizations.

Appendix A

Properties of $\mathrm{SL}(2,\mathbb{N})$ Matrices

Symbol $\mathrm{SL}(2,\mathbb{Z})$ denotes set of all integer 2×2 matrices $\mathbf{A}$ such that $\det \mathbf{A} = +1$. These matrices form a group of transformations.

If we restrict the definition to non-negative integer matrices, we get a monoid (see below):

Definition A.1. The symbol $\mathrm{SL}(2,\mathbb{N})$ denotes set of all non-negative integer matrices $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{N}^{2,2}$ such that $\det \mathbf{A} = ad - bc = 1$.

Convention. The matrices are indexed from zero, hence we write

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} \\ \mathbf{A}_{10} & \mathbf{A}_{11} \end{pmatrix}.$$

Definition A.2. We define matrices $\mathbf{L}, \mathbf{R}, \mathbf{I} \in \mathrm{SL}(2,\mathbb{N})$ as follows:

$$\mathbf{L} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{R} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example. Inverse matrices to $\mathbf{L}, \mathbf{R}$ are

$$\mathbf{L}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{R}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We can see that $\mathbf{L}^{-1}, \mathbf{R}^{-1} \notin \mathrm{SL}(2,\mathbb{N})$, which proves that $\mathrm{SL}(2,\mathbb{N})$ is not a group.

Lemma A.3. *Let $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{N})$ be a matrix. Then there exists unique factorization by matrices $\mathbf{L}$ and $\mathbf{R}$, which are defined before.*

Proof. We will discuss the following 4 cases:

$(a > b \wedge c < d)$ Since $a, b, c, d \in \mathbb{N}$ we may write $a \geq b+1$ and $d \geq c+1$ and estimate $\det \mathbf{A} = ad - bc \geq (b+1)(c+1) - bc = b + c + 1$. From whence it follows that $b = c = 0$ to satisfy $\det \mathbf{A} = 1$ and $\mathbf{A} = \mathbf{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$(a < b \wedge c > d)$ Equivalently $a \leq b - 1$ and $c - 1 \geq d$. And as $a, d \geq 0$, we have $b, c \geq 1$. Hence $\det \mathbf{A} = ad - bc \leq (b-1)(c-1) - bc = 1 - b - c \leq -1$, so $\det \mathbf{A} \neq 1$. This case cannot occur.

$(a \leq b \wedge c \leq d)$ Then the matrix $\mathbf{B} \coloneqq \left(\begin{smallmatrix} a, & b-a \\ c, & d-c \end{smallmatrix}\right)$ is non-negative and $\mathbf{A} = \mathbf{BL}$. Since $\det \mathbf{A} = \det \mathbf{L} = 1$, we have $\det \mathbf{B} = 1$ and thus elements a and c are not simultaneously 0. Now clearly sum of elements of $\mathbf{B}$ is strictly smaller than sum of elements of $\mathbf{A}$.

$(a \geq b \wedge c \geq d)$ Symmetrically to previous case we put $\mathbf{B} \coloneqq \left(\begin{smallmatrix} a-b, & b \\ c-d, & d \end{smallmatrix}\right)$ and get $\mathbf{A} = \mathbf{BR}$.

In each step we diminish the sum of elements of the matrix, thus after finite number of steps we obtain the matrix $\mathbf{I}$.

Finally we need to explain why the factorization is unique. Let us suppose that both cases occur. Thus $a = b$ and $c = d$, which means that $\det \mathbf{A} = 0$, forming contradiction. □

Corollary A.4. *For every* $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{N})$ *either* $(a-b)(c-d) \geq 0$ *or* $\mathbf{A} = \mathbf{I}$.

Lemma A.5. *Let* $\mathbf{A}, \mathbf{A}' \in \mathrm{SL}(2,\mathbb{N})$ *be two matrices. Define for any matrix* $\mathbf{G} \in \mathbb{N}^{2,2}$

$$\begin{aligned} p_{\mathbf{G}} &\coloneqq \mathbf{G}_{00} + \mathbf{G}_{01} && \textit{(sum of top row of matrix)}, \\ q_{\mathbf{G}} &\coloneqq \mathbf{G}_{10} + \mathbf{G}_{11} && \textit{(sum of bottom row of matrix)} \\ \textit{and} \quad N_{\mathbf{G}} &\coloneqq p_{\mathbf{G}} + q_{\mathbf{G}} && \textit{(sum of all elements of matrix)}. \end{aligned}$$

Then

$$p_{\mathbf{A}} = p_{\mathbf{A}'} \ \wedge \ N_{\mathbf{A}} = N_{\mathbf{A}'} \implies \mathbf{A} = \mathbf{A}'.$$

PROOF. We can easily see that if $p_{\mathbf{A}} = p_{\mathbf{A}'}$ and $N_{\mathbf{A}} = N_{\mathbf{A}'}$, then $q_{\mathbf{A}} = q_{\mathbf{A}'}$.

From the definition, it is clear that

$$\begin{aligned} \begin{pmatrix} p_{\mathbf{G}} \\ q_{\mathbf{G}} \end{pmatrix} &= \mathbf{G} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \text{thus} \quad \begin{pmatrix} p_{\mathbf{GH}} \\ q_{\mathbf{GH}} \end{pmatrix} &= \mathbf{G} \cdot \begin{pmatrix} p_{\mathbf{H}} \\ q_{\mathbf{H}} \end{pmatrix}. \end{aligned}$$

Let us suppose for contradiction that $\mathbf{A} \neq \mathbf{A}'$. Factorize the matrices $\mathbf{A}$, $\mathbf{A}'$ according to Lemma A.3, and denote $\mathbf{B}$ the longest common prefix of the factorization, hence, without the loss of generality, exists matrices $\mathbf{C}, \mathbf{C}' \in \mathrm{SL}(2,\mathbb{N})$ such that $\mathbf{A} = \mathbf{BLC}$ and $\mathbf{A}' = \mathbf{BRC}'$. (We should explain why not $\mathbf{B} = \mathbf{A}$ or $\mathbf{B} = \mathbf{A}'$. But if for instance $\mathbf{B} = \mathbf{A}$, then $\mathbf{A}' = \mathbf{AD}$ for some $\mathbf{D} \in \mathrm{SL}(2,\mathbb{N})$, $\mathbf{D} \neq \mathbf{I}$, thus $N_{\mathbf{A}'} > N_{\mathbf{A}}$.)

Matrix $\mathbf{B}$ is regular, hence

$$\begin{aligned}\mathbf{L}\begin{pmatrix}p_{\mathbf{C}}\\ q_{\mathbf{C}}\end{pmatrix} &= \begin{pmatrix}p_{\mathbf{LC}}\\ q_{\mathbf{LC}}\end{pmatrix} = \begin{pmatrix}p_{\mathbf{B}^{-1}\mathbf{A}}\\ q_{\mathbf{B}^{-1}\mathbf{A}}\end{pmatrix} = \mathbf{B}^{-1}\begin{pmatrix}p_{\mathbf{A}}\\ q_{\mathbf{A}}\end{pmatrix}\\ &= \mathbf{B}^{-1}\begin{pmatrix}p_{\mathbf{A}'}\\ q_{\mathbf{A}'}\end{pmatrix} = \begin{pmatrix}p_{\mathbf{B}^{-1}\mathbf{A}'}\\ q_{\mathbf{B}^{-1}\mathbf{A}'}\end{pmatrix} = \begin{pmatrix}p_{\mathbf{RC}'}\\ q_{\mathbf{RC}'}\end{pmatrix} = \mathbf{R}\begin{pmatrix}p_{\mathbf{C}'}\\ q_{\mathbf{C}'}\end{pmatrix} =: \begin{pmatrix}P\\ Q\end{pmatrix}.\end{aligned}$$

Discuss the following cases.

$(P \geq Q)$ Then $\binom{p_{\mathbf{C}}}{q_{\mathbf{C}}} = \mathbf{L}^{-1}\binom{P}{Q} = \binom{P}{Q-P}$, where $Q - P \leq 0$. That is in contradiction with $\mathbf{C} \in \mathrm{SL}(2,\mathbb{N})$, because it should be regular and all its elements should be non-negative.

$(P < Q)$ Then $\binom{p_{\mathbf{C}'}}{q_{\mathbf{C}'}} = \mathbf{R}^{-1}\binom{P}{Q} = \binom{P-Q}{Q}$, where $P - Q < 0$. That is in contradiction with $\mathbf{C}' \in \mathrm{SL}(2,\mathbb{N})$. □

Property A.6. *The set* $\mathrm{SL}(2,\mathbb{N})$ *is a monoid of words over alphabet* $\{\mathbf{L},\mathbf{R}\}$ *with the empty word* $\mathbf{I}$.

The claim needs to be interpreted correctly. We say that there exists isomorphism between $\mathrm{SL}(2,\mathbb{N})$ and between words over two-letter alphabet. That means that $\mathrm{SL}(2,\mathbb{N})$ is closed to multiplication, generated by $\{\mathbf{L},\mathbf{R}\}$ and two matrices are equal, if and only if, their factorizations are equal.

Proof. Take $\mathbf{A},\mathbf{B} \in \mathrm{SL}(2,\mathbb{N})$. Then determinant of $\mathbf{AB}$ is clearly one, and elements of $\mathbf{AB}$ are values of non-negative integer combinations of elements of $\mathbf{A}$ and $\mathbf{B}$, hence non-negative integers.

The rest of claim is already proved as Lemma A.3. □

Theorem A.7. *The mapping* $\mathbf{A} \longleftrightarrow \binom{p_{\mathbf{A}}}{q_{\mathbf{A}}}$ *is one-to-one mapping between* $\mathrm{SL}(2,\mathbb{N})$ *and the set of coprime pairs in* $\mathbb{N}$*, i.e.:*

(1) *for every* $\mathbf{A} \in \mathrm{SL}(2,\mathbb{N})$ *the numbers* $p_{\mathbf{A}}$ *and* $q_{\mathbf{A}}$ *are coprime;*

(2) *for every* $p, q \in \mathbb{N}$ *coprime there exists exactly one* $\mathbf{A} \in \mathrm{SL}(2,\mathbb{N})$ *such that* $p = p_{\mathbf{A}}$ *and* $q = q_{\mathbf{A}}$.

Proof. **(1)** For $\mathbf{A} = \mathbf{I}$ holds that $p_{\mathbf{A}} = q_{\mathbf{A}} = 1$, and number one is coprime with itself.

Suppose p, q coprime. Then $p, p+q$ are coprime, and as well $p+q, q$ are coprime, while

$$\begin{pmatrix}p\\ p+q\end{pmatrix} = \mathbf{L}\begin{pmatrix}p\\ q\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}p+q\\ q\end{pmatrix} = \mathbf{R}\begin{pmatrix}p\\ q\end{pmatrix}.$$

Mathematical induction in length of factorization easily leads to the claim.

(2) The following iterative method is described. Put $p_0 \coloneqq p$, $q_0 \coloneqq q$, $\mathbf{A}_0 = \mathbf{I}$.

If $p_k > q_k$, we put $p_{k+1} \coloneqq p_k - q_k$ and $q_{k+1} \coloneqq q_k$. Follows $\binom{p_{k+1}}{q_{k+1}} = \mathbf{R}\binom{p_k}{q_k}$, hence we put $\mathbf{A}_{k+1} \coloneqq \mathbf{R}\mathbf{A}_k$.

If $p_k < q_k$, we put $p_{k+1} \coloneqq p_k$ and $q_{k+1} \coloneqq q_k - p_k$. Follows $\binom{p_{k+1}}{q_{k+1}} = \mathbf{L}\binom{p_k}{q_k}$, hence we put $\mathbf{A}_{k+1} \coloneqq \mathbf{L}\mathbf{A}_k$.

From the way we construct sequence $\mathbf{A}_k$, it is clear that $\mathbf{A}_k\binom{p}{q} = \mathbf{A}_0\binom{p}{q} = \binom{p}{q}$. And from the fact that p, q are coprime, we know that p_k, q_k are coprime for all k.

If $p_k = q_k$, we end the algorithm. Numbers p_k, q_k are coprime, hence necessarily $\binom{p_k}{q_k} = \binom{1}{1} = \binom{p_{\mathbf{I}}}{q_{\mathbf{I}}}$ and

$$\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{A}_k \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \mathbf{A}_k \begin{pmatrix} p_{\mathbf{I}} \\ q_{\mathbf{I}} \end{pmatrix} = \begin{pmatrix} p_{\mathbf{A}_k} \\ q_{\mathbf{A}_k} \end{pmatrix}.$$

Putting $\mathbf{A} \coloneqq \mathbf{A}_k$ gives the claim. (The uniqueness of matrix $\mathbf{A}$ is shown in Lemma A.5.) $\square$

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